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Distribution of $\omega(n)$ over h-free and h-full numbers

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Let $\omega(n)$ denote the number of distinct prime factors of a natural number n. In 1917, Hardy and Ramanujan proved that $\omega(n)$ has normal order $\log\log n$ over naturals. In this work, we establish the first and second moments of $\omega(n)$ over h-free and h-full numbers using a new counting argument and prove that $\omega(n)$ has normal order $\log\log n$ over these subsets.

Keywords: Prime factors; the ω-function; h-free numbers; h-full numbers; normal order.

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1. Introduction

For a natural number n, let the prime factorization of n be given as

$$n = p_1^{s_1} \dots p_r^{s_r}, \tag{1.1}$$

where p_is are its distinct prime factors and s_is are their respective multiplicities. Let $\Omega(n)$ denote the total number of prime factors in the factorization of n and $\omega(n)$ denote the total number of distinct prime factors in the factorization of n. Hence, $\Omega(n) = \sum_{i=1}^{r} s_i$ and $\omega(n) = r$. Let B_1 be the Mertens constant given by

$$B_1 = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right),$$

with $\gamma \approx 0.57722$, the Euler–Mascheroni constant, and where the sum runs over all primes p. Let

$$B_2 = B_1 + \sum_p \frac{1}{p(p-1)}.$$

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The following average value formulas are well known (see [7, Theorem 430; 4, Sec. 1.4.4]):

$$\sum_{n\leq x}\omega(n) = x\log\log x + B_1 x + O\left(\frac{x}{\log x}\right) \tag{1.2}$$

and

$$\sum_{n\leq x}\Omega(n) = x\log\log x + B_2 x + O\left(\frac{x}{\log x}\right). \tag{1.3}$$

Let $h \geq 2$ be an integer. Let n be a natural number with the factorization given in (1.1). We say n is *h-free* if $s_i \leq h-1$ for all $i \in \{1,\ldots,r\}$, and n is *h-full* if $s_i \geq h$ for all $i \in \{1,\ldots,r\}$. Let $\mathcal{S}_h$ denote the set of h-free numbers and $\mathcal{N}_h$ denote the set of h-full numbers. Let $\gamma_{0,h}$ be the constant defined as

$$\gamma_{0,h} := \prod_p \left(1 + \frac{p - p^{1/h}}{p^2(p^{1/h}-1)}\right), \tag{1.4}$$

where the product runs over all primes p, and B_3 be the constant defined as

$$B_3 := h(B_2 - \log h) + \sum_p \frac{(h+1)p^{1+1/h} - hp - 2hp^{2/h} + (2h-1)p^{1/h}}{(p-1)(p^{1/h}-1)(p^{1+1/h}+p^{1/h}-p)}. \tag{1.5}$$

In [9, Theorems 1 and 2], Jakimczuk and Lalín established the first moments of $\Omega(n)$ over h-free and h-full numbers, respectively, as

$$\sum_{\substack{n\leq x\\ n\in\mathcal{S}_h}}\Omega(n) = \frac{1}{\zeta(h)}x\log\log x + O_h(x) \tag{1.6}$$

and

$$\sum_{\substack{n\leq x\\ n\in\mathcal{N}_h}}\Omega(n) = h\gamma_{0,h}x^{1/h}\log\log x + B_3\gamma_{0,h}x^{1/h} + O_h\left(\frac{x^{1/h}}{\sqrt{\log x}}\right), \tag{1.7}$$

where $\zeta(s)$ represents the classical Riemann ζ-function, and where O_X denotes that the implied big-O constant depends on the variable set X. To obtain (1.6), they used the generating Dirichlet series for h-free numbers. For (1.7), they used the completely additive property of $\Omega(n)$ to split the sum into sums over divisors of n distinguished by their multiplicity, and then used generating series arguments to estimate these sums. This method for the h-full case cannot be extended to the study of $\omega(n)$, since $\omega(n)$ is not completely additive. This limitation of their approach is presented in [11, Line 15, p. 33] where the authors established the function field analog of [9]. Thus, to study the moments of $\omega(n)$ over h-full numbers, we use a new counting argument. At its core, our method relies on counting the natural numbers divisible by a given set of prime powers in a bounded range. This new counting argument not only establishes the average distribution for $\omega(n)$ over h-free and h-full numbers, but also improves the results from [9]. In particular, it makes the coefficient of x in (1.6) explicit and improves the error term in (1.7) to $x^{1/h}/\log x$. Additionally, we employ this argument to establish the second moment estimate for $\omega(n)$ over the sets of h-free and h-full numbers.

The first part of our work employs the counting argument to study the moments of $\omega(n)$ over the subsets of h-free and h-full numbers. As the evidence in (1.2) and (1.3) suggests, the average distribution of $\omega(n)$ differs from that of $\Omega(n)$ in the second main term. Thus, asymptotically, they behave the same. We verify this phenomenon when restricted to these subsets.

We define the constants

$$C_1 := B_1 - \sum_p \frac{p-1}{p(p^h-1)} \tag{1.8}$$

and

$$C_2 := C_1^2 + C_1 - \zeta(2) - \sum_p \left(\frac{p^{h-1}-1}{p^h-1}\right)^2.$$

Using the above definitions, we prove the first and second moments of $\omega(n)$ over h-free numbers as follows.

Theorem 1.1. *Let $x > 2$ be a real number. Let $h \geq 2$ be an integer. Let $\mathcal{S}_h(x)$ be the set of h-free numbers less than or equal to x. Then, we have*

$$\sum_{n \in \mathcal{S}_h(x)} \omega(n) = \frac{1}{\zeta(h)} x \log\log x + \frac{C_1}{\zeta(h)} x + O_h\left(\frac{x}{\log x}\right)$$

and

$$\sum_{n \in \mathcal{S}_h(x)} \omega^2(n) = \frac{1}{\zeta(h)} x (\log\log x)^2 + \frac{2C_1+1}{\zeta(h)} x \log\log x + \frac{C_2}{\zeta(h)} x + O_h\left(\frac{x}{\log x}\right).$$

Let $\mathcal{L}_h(r)$ be the convergent sum defined for $r > h$ as

$$\mathcal{L}_h(r) := \sum_p \frac{1}{p^{(r/h)-1}\left(p - p^{1-1/h} + 1\right)}, \tag{1.9}$$

where the sum runs over all primes. Using this, we define two new constants,

$$D_1 := B_1 - \log h + \mathcal{L}_h(h+1) - \mathcal{L}_h(2h) \tag{1.10}$$

and

$$D_2 := D_1^2 + D_1 - \zeta(2) - \sum_p \left(\frac{1}{p - p^{1-1/h} + 1}\right)^2. \tag{1.11}$$

For the distributions of $\omega(n)$ over h-full numbers, we prove the following.

Theorem 1.2. *Let $x > 2$ be a real number. Let $h \geq 2$ be an integer. Let $\mathcal{N}_h(x)$ be the set of h-full numbers less than or equal to x. We have*

$$\sum_{n \in \mathcal{N}_h(x)} \omega(n) = \gamma_{0,h} x^{1/h} \log\log x + D_1 \gamma_{0,h} x^{1/h} + O_h\left(\frac{x^{1/h}}{\log x}\right)$$

and

$$\sum_{n \in \mathcal{N}_h(x)} \omega^2(n) = \gamma_{0,h} x^{1/h} (\log\log x)^2 + (2D_1 + 1)\, \gamma_{0,h} x^{1/h} \log\log x$$
$$+ D_2 \gamma_{0,h} x^{1/h} + O_h\left(\frac{x^{1/h} \log\log x}{\log x}\right).$$

Next, we introduce the definition of the normal order of an arithmetic function over a subset of natural numbers. This definition is inspired by the work of Elma and Liu [3] over naturals, where they extended the definition of normal order using an increasing function (see [6]) to a definition involving a nondecreasing function. Let $S \subseteq \mathbb{N}$ and $S(x)$ denote the set of natural numbers belonging to S and less than or equal to x. Let $|S(x)|$ denote the cardinality of $S(x)$. Let $f, F : S \to \mathbb{R}_{\geq 0}$ be two functions such that F is nondecreasing. Then, $f(n)$ is said to have normal order $F(n)$ over S if for any $\epsilon > 0$, the number of $n \in S(x)$ that do not satisfy the inequality

$$(1 - \epsilon) F(n) \leq f(n) \leq (1 + \epsilon) F(n)$$

is $o(|S(x)|)$ as $x \to \infty$.

The distribution in (1.2) proves that, on average, $\omega(n)$ behaves like $\log\log n$ over rationals. In [6], Hardy and Ramanujan strengthened this result further by proving that $\omega(n)$ has the normal order $\log\log n$ over natural numbers. In [7, Sec. 22.11], one can find another proof of this result using the variance of $\omega(n)$ (see [7, (22.11.7)]).

Note that $\omega(n)$ may exhibit different normal orders over different subsets of $\mathbb{N}$. For instance, $\omega(p) = 1$ for any prime p, and thus $\omega(n)$ has normal order 1 over the set of all primes. However, in this work, we establish that $\omega(n)$ still has the normal order $\log\log n$ over h-free and h-full numbers. Since the set of h-free numbers has a positive density in $\mathbb{N}$, the proof of normal order of $\omega(n)$ over h-free numbers follows from the classical case. In particular, one can establish that for any $\epsilon > 0$, the number of $n \in \mathcal{S}_h(x)$ that do not satisfy the inequality

$$(1 - \epsilon) \log\log n \leq \omega(n) \leq (1 + \epsilon) \log\log n$$

is $o(|\mathcal{S}_h(x)|)$ as $x \to \infty$. On the other hand, the set of h-full numbers has a density zero in $\mathbb{N}$ and thus does not directly follow from the classical result. However, writing an h-full number n as $n = r_0^h r_1$, where r_0 is the product of all distinct prime factors of n with multiplicity h, one can use the classical result on $\omega(r_0)$ to establish the normal order of $\omega(n)$ over h-full numbers. In this work, we provide another proof of this result using the variance of $\omega(n)$ over the h-full numbers. We prove the following theorem.

Theorem 1.3. *Let $h \geq 2$ be an integer. Let D_2 be defined by* (1.11) *and $\gamma_{0,h}$ be defined by* (1.4). *Let $\mathcal{N}_h(x)$ be the set of h-full numbers less than or equal to x. Then, the variance of $\omega(n)$ over the h-full numbers is given by*

$$\sum_{n\in\mathcal{N}_h(x)} (\omega(n) - \log\log n)^2 = \gamma_{0,h}x^{1/h}\log\log x + D_2\gamma_{0,h}x^{1/h} + O_h\left(\frac{x^{1/h}\log\log x}{\log x}\right).$$

Let $g(x)$ be an increasing function such that $g(x) \to \infty$ as $x \to \infty$. Then the set of natural numbers $n \in \mathcal{N}_h(x)$, such that

$$\frac{|\omega(n) - \log\log n|}{\sqrt{\log\log n}} \geq g(x),$$

is $o(|\mathcal{N}_h(x)|)$. As a consequence, $\omega(n)$ has normal order $\log\log n$ *over the h-full numbers.*

The counting argument used for $\omega(n)$ can be employed in the study of the distribution of $\Omega(n)$ and produce improvement to (1.6) and (1.7) as mentioned earlier. Let C_3 and C_4 be two new constants defined as

$$C_3 := B_2 - \sum_p \frac{h}{p^h - 1}$$

and

$$C_4 := C_3^2 + C_3 - \zeta(2) - \sum_p \left(\frac{p^h - hp + h - 1}{(p-1)(p^h - 1)}\right)^2.$$

One can improve (1.6) as follows.

Theorem 1.4. *Let $x > 2$ be a real number. Let $h \geq 2$ be an integer. Let $\mathcal{S}_h(x)$ be the set of h-free numbers less than or equal to x. Then, we have*

$$\sum_{n\in\mathcal{S}_h(x)} \Omega(n) = \frac{1}{\zeta(h)}x\log\log x + \frac{C_3}{\zeta(h)}x + O_h\left(\frac{x}{\log x}\right)$$

and

$$\sum_{n\in\mathcal{S}_h(x)} \Omega^2(n) = \frac{1}{\zeta(h)}x(\log\log x)^2 + \frac{2C_3 + 1}{\zeta(h)}x\log\log x + \frac{C_4}{\zeta(h)}x + O_h\left(\frac{x}{\log x}\right).$$

Recall B_3 from (1.5). Let B_4 be a new constant defined as

$$B_4 := B_3^2 + B_3 - h^2\zeta(2) - \sum_p \left(\frac{h(p^{1/h} - 1) + 1}{(p^{1/h} - 1)(p - p^{1-1/h} + 1)}\right)^2.$$

Adopting this definition, one can improve (1.7) as follows.

Theorem 1.5. *Let $x > 2$ be a real number. Let $h \geq 2$ be an integer. Let $\mathcal{N}_h(x)$ be the set of h-full numbers less than or equal to x. Then, we have*

$$\sum_{n \in \mathcal{N}_h(x)} \Omega(n) = h\gamma_{0,h} x^{1/h} \log\log x + B_3 \gamma_{0,h} x^{1/h} + O_h\left(\frac{x^{1/h}}{\log x}\right)$$

and

$$\sum_{n \in \mathcal{N}_h(x)} \Omega^2(n) = h^2 \gamma_{0,h} x^{1/h} (\log\log x)^2 + (2B_3 + 1) h\gamma_{0,h} x^{1/h} \log\log x + B_4 \gamma_{0,h} x^{1/h} + O_h\left(\frac{x^{1/h} \log\log x}{\log x}\right).$$

Using the last two theorems and the method of variance demonstrated for $\omega(n)$ in this paper, one can establish the normal order of $\Omega(n)$ over h-free and h-full numbers as follows.

Corollary 1.1. $\Omega(n)$ *has normal order* $\log\log n$ *over h-free numbers and normal order* $h \log\log n$ *over h-full numbers.*

The proofs for the $\Omega(n)$ results are not included in this paper. Interested readers can find the complete proofs in the Ph.D. thesis of Das [2].

2. Lemmata

In this section, we list all the lemmas required to study the first and second moments of $\omega(n)$ over the h-free and h-full numbers. First, we recall the following results regarding sums over primes necessary for the study.

Lemma 2.1 ([1, Lemma 1.2]). *Let $\tau > 1$ be any real number. Then,*

$$\sum_{p \geq x} \frac{1}{p^\tau} = \frac{1}{(\tau - 1) x^{\tau-1} (\log x)} + O\left(\frac{1}{x^{\tau-1} (\log x)^\tau}\right).$$

Lemma 2.2. *Let α be any real number satisfying $0 < \alpha < 1$. Then,*

$$\sum_{p \leq x} \frac{1}{p^\alpha} = O_\alpha\left(\frac{x^{1-\alpha}}{\log x}\right).$$

Proof. Recall that the classical prime number theorem yields

$$\pi(x) := \sum_{p \leq x} 1 = \frac{x}{\log x} + O\left(\frac{x}{(\log x)^2}\right). \tag{2.1}$$

Thus, using partial summation and integration by parts, we obtain

$$\sum_{p \leq x} \frac{1}{p^\alpha} = \frac{\pi(x)}{x^\alpha} + \alpha \int_{2^-}^{x} \frac{\pi(t)}{t^{\alpha+1}} dt \ll_\alpha \frac{x^{1-\alpha}}{\log x},$$

which completes the proof. □

Lemma 2.3 ([12, Exercise 9.4.4]). *For $x > 2$, we have*

$$\sum_{p \le x/2} \frac{1}{p \log(x/p)} = O\left(\frac{\log\log x}{\log x}\right).$$

Lemma 2.4. *Let p and q denote prime numbers. For $x > 2$, we have*

$$\sum_{\substack{p,q \\ pq \le x}} \frac{1}{pq} = (\log\log x)^2 + 2B_1 \log\log x + B_1^2 - \zeta(2) + O\left(\frac{\log\log x}{\log x}\right).$$

Proof. The proof follows the exact arguments of Saidak [13, Lemma 3] except for a small correction. We notice that $\int_0^1 \log(1-t)/t \, dt = -\zeta(2)$ which proves the value of $-\zeta(2)$ in the main result as opposed to $+\zeta(2)$ stated by Saidak. $\square$

Finally, we recall the following results regarding the density of a particular sequence of h-free numbers.

Lemma 2.5 ([10, Lemma 3]). *Let $x > 2$ be a real number. Let $h \ge 2$ be an integer. Let $\mathcal{S}_h(x)$ be the set of h-free numbers less than or equal to x. Let $q_1, \dots, q_r$ be the prime numbers. Then, we have*

$$\sum_{\substack{n \in \mathcal{S}_h(x) \\ (q_i, n) = 1, \ \forall i \in \{1, \dots, r\}}} 1 = \prod_{i=1}^{r} \left(\frac{q_i^h - q_i^{h-1}}{q_i^h - 1}\right) \frac{x}{\zeta(h)} + O_h\left(2^r x^{1/h}\right).$$

3. The First and Second Moments of $\omega(n)$ over h-Free Numbers

In this section, we establish the asymptotic distribution of $\omega(n)$ over h-free numbers.

Proof of Theorem 1.1. Let $p^k || n$ denote the property that p^k divides n but p^{k+1} does not divide n. Then writing $n = p^k y$ with $(y, p) = 1$ for such n and using $r = \min\{h-1, \lfloor \log x / \log p \rfloor\}$, we obtain

$$\sum_{n \in \mathcal{S}_h(x)} \omega(n) = \sum_{n \in \mathcal{S}_h(x)} \sum_{\substack{p \\ p | n}} 1 = \sum_{p \le x} \sum_{k=1}^{r} \sum_{\substack{n \in \mathcal{S}_h(x) \\ p^k || n}} 1 = \sum_{p \le x} \sum_{k=1}^{r} \sum_{\substack{y \in \mathcal{S}_h(x/p^k) \\ (y,p) = 1}} 1.$$

Now, first using Lemma 2.5 for a single prime p on the above and then using Lemma 2.2, we obtain

$$\sum_{n \in \mathcal{S}_h(x)} \omega(n) = \sum_{p \le x} \sum_{k=1}^{r} \left(\frac{p^h - p^{h-1}}{p^k (p^h - 1)}\right) \frac{x}{\zeta(h)} + O_h\left(\frac{x}{\log x}\right). \tag{3.1}$$

Using the bound $\lfloor x \rfloor \ge x - 1$ and (2.1), we can write

$$\sum_{p \le x} \sum_{k=1}^{r} \left(\frac{p^h - p^{h-1}}{p^k (p^h - 1)}\right) = \sum_{p \le x} \sum_{k=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k (p^h - 1)}\right) + O\left(\frac{1}{\log x}\right). \tag{3.2}$$

Now, using

$$\sum_{k=1}^{h-1} \frac{p^h - p^{h-1}}{p^k(p^h-1)} = \frac{1}{p} - \frac{p-1}{p(p^h-1)},$$

and Mertens' second theorem given as

$$\sum_{p\le x} \frac{1}{p} = \log\log x + B_1 + O\left(\frac{1}{\log x}\right), \tag{3.3}$$

and using Lemma 2.1 with $\tau = h$, we obtain

$$\sum_{p\le x}\sum_{k=1}^{h-1}\left(\frac{p^h - p^{h-1}}{p^k(p^h-1)}\right) = \log\log x + C_1 + O_h\left(\frac{1}{\log x}\right),$$

where C_1 is defined in (1.8). Combining the above with (3.1) and (3.2) completes the first part of the proof.

Next, let $r_p = \min\{h-1, \lfloor \log x/\log p\rfloor\}$ and $r_q = \min\{h-1, \lfloor \log x/\log q\rfloor\}$. Then, we have

$$\begin{aligned}\sum_{n\in\mathcal{S}_h(x)} \omega^2(n) &= \sum_{n\in\mathcal{S}_h(x)}\left(\sum_{\substack{p\\ p\mid n}} 1\right)^2 \\ &= \sum_{n\in\mathcal{S}_h(x)} \omega(n) + \sum_{n\in\mathcal{S}_h(x)} \sum_{\substack{p,q \\ p^k\|n,\ q^l\|n,\ p\ne q}} \left(\sum_{k=1}^{r_p}\sum_{l=1}^{r_q} 1\right),\end{aligned} \tag{3.4}$$

where p and q above denote the primes. The first sum on the right-hand side is the first moment studied above. For the second sum, we rewrite the sum and use Lemma 2.5 for the two primes p and q to obtain

$$\begin{aligned}&\sum_{n\in\mathcal{S}_h(x)} \sum_{\substack{p,q \\ p^k\|n,\ q^l\|n,\ p\ne q}} \left(\sum_{k=1}^{r_p}\sum_{l=1}^{r_q} 1\right) \\ &= \sum_{\substack{p,q\\ p\ne q,\ pq\le x}} \sum_{k=1}^{r_p}\sum_{l=1}^{r_q} \left(\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)\left(\frac{q^h-q^{h-1}}{q^l(q^h-1)}\right)\frac{x}{\zeta(h)} + O_h\left(\frac{x^{1/h}}{p^{k/h}q^{l/h}}\right)\right).\end{aligned} \tag{3.5}$$

We employ Lemma 2.2 with $\alpha = 1/h$ and Lemma 2.3 to estimate the error term above as

$$\sum_{\substack{p,q\\ p\ne q,\ pq\le x}} \sum_{k=1}^{r_p}\sum_{l=1}^{r_q} \frac{x^{1/h}}{p^{k/h}q^{l/h}} \ll x^{1/h} \sum_{\substack{p,q\\ p\ne q,\ pq\le x}} \frac{1}{(pq)^{1/h}} \ll_h \frac{x\log\log x}{\log x}. \tag{3.6}$$

For estimating the main term in (3.5), we consider the set R defined as

$$R := \left\{ p \leq x \;\middle|\; \left\lfloor \frac{\log x}{\log p} \right\rfloor < h-1 \right\}.$$

Using the definitions of r_p and r_q, we rewrite

$$\begin{aligned}
&\sum_{\substack{p,q \\ p\neq q,\ pq\leq x}} \sum_{k=1}^{r_p}\sum_{l=1}^{r_q} \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)}\right)\left(\frac{q^h - q^{h-1}}{q^l(q^h-1)}\right) \\
&\quad = \sum_{\substack{p,q \\ p\neq q,\ pq\leq x}} \sum_{k=1}^{h-1}\sum_{l=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)}\right)\left(\frac{q^h - q^{h-1}}{q^l(q^h-1)}\right) - I_1 - I_2 + I_3, \qquad (3.7)
\end{aligned}$$

where

$$I_1 = \sum_{\substack{p,q \\ p\neq q,\ pq\leq x \\ p\in R}} \sum_{k=\lfloor \frac{\log x}{\log p}\rfloor+1}^{h-1} \sum_{l=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)}\right)\left(\frac{q^h - q^{h-1}}{q^l(q^h-1)}\right),$$

$$I_2 = \sum_{\substack{p,q \\ p\neq q,\ pq\leq x \\ q\in R}} \sum_{k=1}^{h-1} \sum_{l=\lfloor \frac{\log x}{\log q}\rfloor+1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)}\right)\left(\frac{q^h - q^{h-1}}{q^l(q^h-1)}\right),$$

$$I_3 = \sum_{\substack{p,q \\ p\neq q,\ pq\leq x \\ p,q\in R}} \sum_{k=\lfloor \frac{\log x}{\log p}\rfloor+1}^{h-1} \sum_{l=\lfloor \frac{\log x}{\log q}\rfloor+1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)}\right)\left(\frac{q^h - q^{h-1}}{q^l(q^h-1)}\right).$$

The sums I_1, I_2 and I_3 contribute to the error term. Note that $I_1 = I_2$. In fact, using (3.3) and $\lfloor x\rfloor \geq x-1$, we estimate

$$I_1 \ll \sum_{\substack{p,q \\ p\neq q,\ pq\leq x \\ p\in R}} \frac{1}{p^{\frac{\log x}{\log p}}}\frac{1}{q} \ll \frac{1}{x}\sum_{q\leq x}\frac{1}{q} \ll \frac{\log\log x}{x}.$$

Finally, for I_3, using (2.1), we have

$$I_3 \ll \sum_{\substack{p,q \\ p\neq q,\ pq\leq x \\ p,q\in R}} \frac{1}{p^{\frac{\log x}{\log p}}}\frac{1}{q^{\frac{\log x}{\log q}}} \ll \frac{1}{x^2}\sum_{p\leq x} 1 \ll \frac{1}{x\log x}.$$

We next estimate the main term in (3.7). First, we rewrite the sum as

$$\begin{aligned}
&\sum_{\substack{p,q \\ p\neq q,\ pq\leq x}} \sum_{k=1}^{h-1}\sum_{l=1}^{h-1} \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)}\right)\left(\frac{q^h - q^{h-1}}{q^l(q^h-1)}\right) \\
&\quad = \sum_{\substack{p,q \\ pq\leq x}} \left(\frac{p^{h-1}-1}{p^h-1}\right)\left(\frac{q^{h-1}-1}{q^h-1}\right) - \sum_{\substack{p \\ p\leq x^{1/2}}} \left(\frac{p^{h-1}-1}{p^h-1}\right)^2. \qquad (3.8)
\end{aligned}$$

The second sum above is estimated using Lemma 2.1 as

$$\sum_{\substack{p \\ p\le x^{1/2}}} \left(\frac{p^{h-1}-1}{p^h-1}\right)^2 = \sum_p \left(\frac{p^{h-1}-1}{p^h-1}\right)^2 + O\left(\frac{1}{x^{1/2}\log x}\right). \tag{3.9}$$

For the first sum on the right-hand side in (3.8), using

$$\frac{p^{h-1}-1}{p^h-1} = \frac{1}{p} - \frac{p-1}{p(p^h-1)},$$

and the symmetry in p and q, we have

$$\begin{aligned}
&\sum_{\substack{p,q \\ pq\le x}} \left(\frac{p^{h-1}-1}{p^h-1}\right)\left(\frac{q^{h-1}-1}{q^h-1}\right) \\
&\quad = \sum_{\substack{p,q \\ pq\le x}} \frac{1}{pq} - 2\sum_{\substack{p,q \\ pq\le x}} \frac{1}{p}\left(\frac{q-1}{q(q^h-1)}\right) + \sum_{\substack{p,q \\ pq\le x}} \left(\frac{p-1}{p(p^h-1)}\right)\left(\frac{q-1}{q(q^h-1)}\right).
\end{aligned}$$

We estimate the sums on the right-hand side above separately. For the first sum, we use Lemma 2.4. For the second sum, we use Lemma 2.1 and then (3.3) and (2.1) to obtain

$$\begin{aligned}
\sum_{\substack{p,q \\ pq\le x}} \frac{1}{p}\left(\frac{q-1}{q(q^h-1)}\right) &= \sum_{\substack{p \\ p\le x/2}} \frac{1}{p}\left(\sum_p \frac{p-1}{p(p^h-1)} + O\left(\frac{1}{(x/p)\log(x/p)}\right)\right) \\
&= \left(\sum_p \frac{p-1}{p(p^h-1)}\right)(\log\log x + B_1) + O\left(\frac{1}{\log x}\right).
\end{aligned}$$

For the third sum, we use Lemma 2.1 twice and then Lemma 2.3 to obtain

$$\sum_{\substack{p,q \\ pq\le x}} \left(\frac{p-1}{p(p^h-1)}\right)\left(\frac{q-1}{q(q^h-1)}\right) = \left(\sum_p \frac{p-1}{p(p^h-1)}\right)^2 + O\left(\frac{\log\log x}{x\log x}\right).$$

Combining the last three results and Lemma 2.4, we obtain

$$\begin{aligned}
\sum_{\substack{p,q \\ pq\le x}} \left(\frac{p^{h-1}-1}{p^h-1}\right)\left(\frac{q^{h-1}-1}{q^h-1}\right) &= (\log\log x)^2 + 2C_1\log\log x + C_1^2 - \zeta(2) \\
&\quad + O\left(\frac{\log\log x}{\log x}\right).
\end{aligned}$$

Combining (3.4)–(3.9) with the above equation and using the first-moment estimate, we obtain the required second moment. □

4. The First and Second Moments of $\omega(n)$ over h-Full Numbers

Let $h \geq 2$ be an integer. Recall that $\mathcal{N}_h$ is the set of h-full numbers and $\mathcal{N}_h(x)$ is the set of h-full numbers less than or equal to x. Note that if $n \in \mathcal{N}_h(x)$, then for any prime $p|n$, we must have $p^h|n$. Moreover, the maximal power of p that divides any such $n \leq x$ is $\lfloor \log x/\log p \rfloor$. Using this and $n = p^k y$ with $(y,p) = 1$ when $p^k || n$, we can write

$$\sum_{n\in\mathcal{N}_h(x)} \omega(n) = \sum_{n\in\mathcal{N}_h(x)} \sum_{\substack{p \\ p|n}} 1 = \sum_{p\leq x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p} \rfloor} \sum_{\substack{y\in\mathcal{N}_h(x/p^k) \\ (y,p)=1}} 1. \tag{4.1}$$

Let q be a fixed prime. We intend to estimate the sum

$$A_{q,h}(y) = \sum_{\substack{n\in\mathcal{N}_h(y) \\ (n,q)=1}} 1. \tag{4.2}$$

To prove the estimate for the above sum, we will use the techniques from the work of Ivić and Shiu [8]. Let's recall some of the results about h-full numbers from their work in the following.

Notice that the generating series for the h-full numbers is defined on $\Re(s) > 1/h$ as

$$N_h(s) = \sum_{n\in\mathcal{N}_h} \frac{1}{n^s} = \prod_p \left(1 + \frac{1}{p^{hs}} + \cdots + \frac{1}{p^{ks}} + \cdots\right) = \prod_p \left(1 + \frac{p^{-hs}}{1-p^{-s}}\right).$$

Let $A_h(y)$ denote the number of h-full integers not exceeding y. In [8, Sec. 1], Ivić and Shiu proved that

$$N_h(s) = \zeta(hs)\zeta((h+1)s)\ldots\zeta((2h-1)s)\zeta^{-1}((2h+2)s)\phi_h(s), \tag{4.3}$$

where $\phi_h(s)$ satisfies

$$\prod_p \left(1 - p^{-(2h+2)s} + \sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h} p^{-rs}\right) = \zeta^{-1}((2h+2)s)\phi_h(s), \tag{4.4}$$

and where $a_{r,h}$ satisfies the identity

$$\left(1 + \frac{v^h}{1-v}\right)(1-v^h)\ldots(1-v^{2h-1}) = 1 - v^{2h+2} + \sum_{2h+3}^{(3h^2+h-2)/2} a_{r,h} v^r. \tag{4.5}$$

Note that $|a_{r,h}| \leq h2^h$, $\phi_2(s) = 1$ and $\phi_h(s)$ has a Dirichlet series with abscissa of absolute convergence equal to $1/(2h+3)$ if $h > 2$. Moreover, it is also established that

$$N_h(s) = G_h(s)K_h(s),$$

where

$$K_h(s) = \sum_{n=1}^{\infty} k_h(n)n^{-s} = \zeta(hs)\zeta((h+1)s)\ldots\zeta((2h-1)s) \tag{4.6}$$

and

$$G_h(s) = \sum_{n=1}^{\infty} g_h(n)n^{-s} = \frac{\phi_h(s)}{\zeta((2h+2)s)}$$

and where $G_h(s)$ converges absolutely in $\Re(s) > 1/(2h+2)$. Thus, one can write

$$A_h(y) = \sum_{mn \le y} g_h(m)k_h(n).$$

Additionally, let $S_h(y)$ denote the sum

$$S_h(y) := \sum_{n \le y} k_h(n). \tag{4.7}$$

In [8, Theorem 1], Ivić and Shiu proved that

$$S_h(y) = \sum_{r=h}^{2h-1} C_{r,h} y^{1/r} + \Delta_h^*(y), \tag{4.8}$$

where

$$C_{r,h} = \prod_{j=h, j \neq r}^{2h-1} \zeta(j/r) \tag{4.9}$$

and

$$\Delta_h^*(y) \ll y^{\eta_h}, \tag{4.10}$$

with $1/(2h+2) < \eta_h < 1/(2h-1)$. Using the above prerequisite results, we establish an asymptotic result for $A_{q,h}(y)$ as defined in (4.2). We prove the following lemma.

Lemma 4.1. *Let $q_1, q_2, \ldots, q_r$ be distinct primes. Let*

$$A_{q_1,\ldots,q_r,h}(x) := \sum_{\substack{n \in \mathcal{N}_h(x) \\ (n,q_1)=\cdots=(n,q_r)=1}} 1.$$

For any $x > 2$, we have

$$A_{q_1,\ldots,q_r,h}(x) = \gamma_{q_1,\ldots,q_r,0,h} x^{\frac{1}{h}} + \gamma_{q_1,\ldots,q_r,1,h} x^{\frac{1}{h+1}} + \cdots + \gamma_{q_1,\ldots,q_r,h-1,h} x^{\frac{1}{2h-1}} + O_h(x^{\eta_h}),$$

where $1/(2h+2) < \eta_h < 1/(2h-1)$, and for $i \in \{0, 1, \ldots, h-1\}$,

$$\gamma_{q_1,\ldots,q_r,i,h} = C_{h+i,h} \frac{\phi_h(1/(h+i))}{\zeta((2h+2)/(h+i)) \left(\prod_{j=1}^{r} \left(1 + \frac{q_j^{-h/(h+i)}}{1-q_j^{-1/(h+i)}} \right) \right)}, \tag{4.11}$$

with $C_{h+i,h}$ defined in (4.9) and $\phi_h(s)$ satisfying (4.4).

Proof. Let us consider the sum defined on $\Re(s) > 1/h$ as

$$N_{q_1,\ldots,q_r,h}(s) = \sum_{\substack{n\in\mathcal{N}_h \\ (n,q_1)=\cdots=(n,q_r)=1}} \frac{1}{n^s} = \prod_{\substack{p \\ p\neq q_1,\ldots,q_r}} \left(1 + \frac{p^{-hs}}{1-p^{-s}}\right).$$

By (4.3), we have

$$N_{q_1,\ldots,q_r,h}(s) = \zeta(hs)\zeta((h+1)s)\ldots\zeta((2h-1)s)\frac{\zeta^{-1}((2h+2)s)\phi_h(s)}{\prod_{j=1}^r\left(1+\frac{q_j^{-hs}}{1-q_j^{-s}}\right)}.$$

Thus, we can write

$$N_{q_1,\ldots,q_r,h}(s) = G_{q_1,\ldots,q_r,h}(s)K_h(s),$$

where $K_h(s)$ is defined in (4.6) and

$$G_{q,h}(s) = \sum_{n=1}^{\infty} g_{q_1,\ldots,q_r,h}(n)n^{-s} = \frac{\phi_h(s)}{\zeta((2h+2)s)\left(\prod_{j=1}^r\left(1+\frac{q_j^{-hs}}{1-q_j^{-s}}\right)\right)}.$$

By (4.4) and (4.5), we have

$$\frac{\phi_h(s)}{\zeta((2h+2)s)\left(\prod_{j=1}^r\left(1+\frac{q_j^{-hs}}{1-q_j^{-s}}\right)\right)} = \prod_{\substack{p \\ p\neq q_1,\ldots,q_r}}\left(1-p^{-(2h+2)s}+\sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h}p^{-rs}\right)\left(\prod_{j=1}^{r}\prod_{r=h}^{2h-1}\left(1-q_j^{-rs}\right)\right). \tag{4.12}$$

Note that, for $\Re(s) > 1/(2h+2)$ and for all sufficiently large p, using $|a_{r,h}| \leq h2^h$, we have

$$\left|p^{-(2h+2)s}+\sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h}p^{-rs}\right| < 1.$$

Moreover, $|q_j^{-rs}| < 1$ for all $j = 1,\ldots,r$ and for all $r \in \{h, h+1,\ldots,2h-1\}$. Thus, taking the logarithm of the right-hand side of (4.12) in the region $\Re(s) > 1/(2h+2)$

and using $|a_{r,h}| \leq h2^h$ again, we obtain

$$\log\left(\prod_{\substack{p \\ p\neq q_1,\ldots,q_r}}\left(1-p^{-(2h+2)s}+\sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h}p^{-rs}\right)\left(\prod_{j=1}^{r}\prod_{r=h}^{2h-1}\left(1-q_j^{-rs}\right)\right)\right)$$

$$= \sum_{\substack{p \\ p\neq q_1,\ldots,q_r}}\log\left(1-p^{-(2h+2)s}+\sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h}p^{-rs}\right)+\sum_{j=1}^{r}\sum_{r=h}^{2h-1}\log\left(1-q_j^{-rs}\right)$$

$$\ll_{h,r} \sum_p \frac{1}{p^{(2h+2)\Re(s)}}+\frac{1}{2^{h\Re(s)}}.$$

The last term of the previous equation is bounded with $\ll_h \zeta((2h+2)\Re(s))+1/2^{h\Re(s)}$ and thus is finite for $\Re(s) > 1/(2h+2)$. Hence, by the theory of convergence of infinite products, $G_{q,h}(s)$ converges absolutely in $\Re(s) > 1/(2h+2)$.

Now, we can write

$$A_{q_1,\ldots,q_r,h}(x) = \sum_{mn\leq x} g_{q_1,\ldots,q_r,h}(m)k_h(n) = \sum_{m\leq x} g_{q_1,\ldots,q_r,h}(m)S_h(x/m),$$

where S_h is defined in (4.7). Further, using (4.8), we obtain

$$A_{q_1,\ldots,q_r,h}(x) = \sum_{r=h}^{2h-1} C_{r,h}x^{1/r}\left(\sum_{m\leq x}\frac{g_{q_1,\ldots,q_r,h}(m)}{m^{1/r}}\right)+\sum_{m\leq x} g_{q_1,\ldots,q_r,h}(m)\Delta_h^*(x/m).$$

We need to estimate the sums above. Note that

$$\sum_{n\leq x} g_{q_1,\ldots,q_r,h}(n)n^{-\eta_h} = O_h(1).$$

Thus, for each $r = h, h+1, \ldots, 2h-1$, using partial summation, we obtain

$$\sum_{m\leq x}\frac{g_{q_1,\ldots,q_r,h}(m)}{m^{1/r}} = G_{q_1,\ldots,q_r,h}(1/r) - \sum_{m>x}\frac{g_{q_1,\ldots,q_r,h}(m)}{m^{1/r}}$$

$$= G_{q_1,\ldots,q_r,h}(1/r) + O_h\left(x^{\eta_h-\frac{1}{r}}\right),$$

and by (4.10),

$$\sum_{m\leq x} g_{q_1,\ldots,q_r,h}(m)\Delta_h^*(x/m) \ll x^{\eta_h}\sum_{m\leq x}|g_{q_1,\ldots,q_r,h}(m)|m^{-\eta_h} \ll_h x^{\eta_h}.$$

Combining the above results, we obtain

$$A_{q_1,\ldots,q_r,h}(x) = \sum_{r=h}^{2h-1} C_{r,h} G_{q_1,\ldots,q_r,h}(1/r) x^{1/r} + O_h\left(x^{\eta_h}\right)$$
$$= \gamma_{q_1,\ldots,q_r,0,h} x^{1/h} + \cdots + \gamma_{q_1,\ldots,q_r,h-1,h} x^{1/(2h-1)} + O_h\left(x^{\eta_h}\right),$$

which completes the proof. □

Remark 4.1. Note that

$$\gamma_{q_1,\ldots,q_r,0,h} = \frac{\gamma_{0,h}}{\prod_{j=1}^{r}\left(1 + \frac{q_j^{-1}}{1-q_j^{-1/h}}\right)}, \tag{4.13}$$

where

$$\gamma_{0,h} = C_{h,h} \frac{\phi_h(1/h)}{\zeta((2h+2)/h)} = \prod_p \left(1 + \frac{p - p^{1/h}}{p^2(p^{1/h} - 1)}\right), \tag{4.14}$$

with the product formula described in [9, p. 3].

We use Lemma 4.1 with one prime to prove the first and second moment estimates for $\omega(n)$ over the h-full numbers. We prove the following.

Proof of Theorem 1.2. Inserting the formula for $A_{q,h}(y)$ given in Lemma 4.1 with $y = x/p^k$ in (4.1), we obtain

$$\sum_{n\in\mathcal{N}_h(x)} \omega(n)$$
$$= \sum_{p\le x^{1/h}} \sum_{k=h}^{\left\lfloor \frac{\log x}{\log p}\right\rfloor} A_{p,h}(x/p^k)$$
$$= \sum_{p\le x^{1/h}} \sum_{k=h}^{\left\lfloor \frac{\log x}{\log p}\right\rfloor} \left(\gamma_{p,0,h}(x/p^k)^{1/h} + \cdots + \gamma_{p,h-1,h}(x/p^k)^{1/(2h-1)} + O_h((x/p^k)^{\eta})\right). \tag{4.15}$$

Let us study the main term above given as

$$\sum_{p\le x^{1/h}} \sum_{k=h}^{\left\lfloor \frac{\log x}{\log p}\right\rfloor} \gamma_{p,0,h}(x/p^k)^{1/h} = \gamma_{0,h} x^{1/h} \sum_{p\le x^{1/h}} \sum_{k=h}^{\left\lfloor \frac{\log x}{\log p}\right\rfloor} \left(\frac{1}{p^{k/h}\left(1 + \frac{p^{-1}}{1-p^{-1/h}}\right)}\right).$$

We obtain

$$\begin{aligned}\gamma_{0,h}x^{1/h}\sum_{p\le x^{1/h}}\sum_{k=h}^{\left\lfloor\frac{\log x}{\log p}\right\rfloor}\left(\frac{1}{p^{k/h}\left(1+\frac{p^{-1}}{1-p^{-1/h}}\right)}\right)\\ =\gamma_{0,h}x^{1/h}\sum_{p\le x^{1/h}}\frac{1}{p\left(1-p^{-1/h}+p^{-1}\right)}-\gamma_{0,h}x^{1/h}\sum_{p\le x^{1/h}}\frac{(p^{-1/h})^{\left\lfloor\frac{\log x}{\log p}\right\rfloor-h+1}}{p\left(1-p^{-1/h}+p^{-1}\right)}.\end{aligned} \tag{4.16}$$

Using $\lfloor x\rfloor\ge x-1$ and then (2.1) with $y=x^{1/h}$, we bound the second term above with

$$\gamma_{0,h}x^{1/h}\sum_{p\le x^{1/h}}\frac{(p^{-1/h})^{\left\lfloor\frac{\log x}{\log p}\right\rfloor-h+1}}{p\left(1-p^{-1/h}+p^{-1}\right)}\ll_h\frac{x^{1/h}}{\log x}. \tag{4.17}$$

Thus, it remains to study the first term in (4.16). A little manipulation yields

$$\sum_{p\le x^{1/h}}\frac{1}{p\left(1-p^{-1/h}+p^{-1}\right)}=\sum_{p\le x^{1/h}}\frac{1}{p}+\sum_{p\le x^{1/h}}\frac{p^{-1/h}-p^{-1}}{p\left(1-p^{-1/h}+p^{-1}\right)}. \tag{4.18}$$

For the first sum on the right-hand side above, we use (3.3) to obtain

$$\sum_{p\le x^{1/h}}\frac{1}{p}=\log\log x-\log h+B_1+O_h\left(\frac{1}{\log x}\right). \tag{4.19}$$

For the second sum on the right-hand side of (4.18), we use the convergence of the corresponding infinite series and Lemma 2.1 to obtain

$$\sum_{p\le x^{1/h}}\frac{p^{-1/h}-p^{-1}}{p\left(1-p^{-1/h}+p^{-1}\right)}=\mathcal{L}_h(h+1)-\mathcal{L}_h(2h)+O_h\left(\frac{1}{x^{1/h^2}(\log x)}\right).$$

Combining the last three results, we obtain

$$\sum_{p\le x^{1/h}}\frac{1}{p\left(1-p^{-1/h}+p^{-1}\right)}=\log\log x+D_1+O_h\left(\frac{1}{\log x}\right), \tag{4.20}$$

where D_1 is defined in (1.10). Combining the above result with (4.15)–(4.17), we obtain

$$\sum_{n\in\mathcal{N}_h(x)} \omega(n) = \gamma_{0,h} x^{1/h} \log\log x + D_1 \gamma_{0,h} x^{1/h} + O_h\left(\frac{x^{1/h}}{\log x}\right) + O_h\left(\sum_{p\le x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p}\rfloor} (x/p^k)^{1/(h+1)}\right). \tag{4.21}$$

For the error term above, using Lemma 2.2 with $\alpha = h/(h+1)$, we obtain

$$\sum_{p\le x^{1/h}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p}\rfloor} (x/p^k)^{1/(h+1)} \ll x^{1/(h+1)} \sum_{p\le x^{1/h}} \frac{1}{p^{h/(h+1)}} \ll_h \frac{x^{1/h}}{\log x}.$$

Inserting the above back into (4.21) completes the first part of the proof.

Now, note that

$$\begin{aligned}\sum_{n\in\mathcal{N}_h(x)} \omega^2(n) &= \sum_{n\in\mathcal{N}_h(x)} \left(\sum_{\substack{p\\ p^h|n}} 1\right)^2 \\ &= \sum_{n\in\mathcal{N}_h(x)} \omega(n) + \sum_{n\in\mathcal{N}_h(x)} \sum_{\substack{p,q\\ p^k||n,\ q^l||n,\ p\ne q}} \left(\sum_{k=h}^{\lfloor \frac{\log x}{\log p}\rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q}\rfloor} 1\right). \end{aligned} \tag{4.22}$$

The first sum on the right-hand side above is the first moment studied earlier. For the second sum, we first rewrite the sum, use Lemma 4.1 with two distinct primes p and q and use (4.13) to obtain

$$\begin{aligned}&\sum_{n\in\mathcal{N}_h(x)} \sum_{\substack{p,q\\ p^k||n,\ q^l||n,\ p\ne q}} \left(\sum_{k=h}^{\lfloor \frac{\log x}{\log p}\rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q}\rfloor} 1\right) \\ &= \gamma_{0,h} x^{1/h} \sum_{\substack{p,q\\ p\ne q,\ pq\le x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p}\rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q}\rfloor} \frac{1}{p^{k/h}\left(1+\frac{p^{-1}}{1-p^{-1/h}}\right)} \frac{1}{q^{l/h}\left(1+\frac{q^{-1}}{1-q^{-1/h}}\right)} \\ &\quad + O_h\left(x^{\frac{1}{h+1}} \sum_{\substack{p,q\\ p\ne q,\ pq\le x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p}\rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q}\rfloor} \frac{1}{p^{k/(h+1)} q^{l/(h+1)}}\right). \end{aligned} \tag{4.23}$$

The error term above is bounded by using Lemma 2.2 and then Lemma 2.3 as follows:

$$
\begin{aligned}
x^{\frac{1}{h+1}} & \sum_{\substack{p,q \\ p\neq q,\ pq\le x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p}\rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q}\rfloor} \frac{1}{p^{k/(h+1)} q^{l/(h+1)}} \\
&\ll_h x^{\frac{1}{h+1}} \sum_{\substack{p \\ p\le x^{1/h}/2}} \frac{1}{p^{h/(h+1)}} \sum_{\substack{q \\ q\le x^{1/h}/p}} \frac{1}{q^{h/(h+1)}} \\
&\ll_h x^{\frac{1}{h}} \sum_{\substack{p \\ p\le x^{1/h}/2}} \frac{1}{p\log(x^{1/h}/p)} \\
&\ll_h \frac{x^{\frac{1}{h}} \log\log x}{\log x}. \qquad (4.24)
\end{aligned}
$$

Next, we estimate the main term in (4.23). First note that

$$
\sum_{k=h}^{\lfloor \frac{\log x}{\log p}\rfloor} \frac{1}{p^{k/h}\left(1+\frac{p^{-1}}{1-p^{-1/h}}\right)} = \frac{1}{p(1-p^{-1/h}+p^{-1})} - \frac{p^{-\frac{1}{h}\left(\lfloor \frac{\log x}{\log p}\rfloor - h+1\right)}}{p(1-p^{-1/h}+p^{-1})}.
$$

Thus, using a similar result for a prime q as above and the symmetry of two primes p and q, we deduce

$$
\begin{aligned}
\gamma_{0,h} x^{1/h} & \sum_{\substack{p,q \\ p\neq q,\ pq\le x^{1/h}}} \sum_{k=h}^{\lfloor \frac{\log x}{\log p}\rfloor} \sum_{l=h}^{\lfloor \frac{\log x}{\log q}\rfloor} \frac{1}{p^{k/h}\left(1+\frac{p^{-1}}{1-p^{-1/h}}\right)} \frac{1}{q^{l/h}\left(1+\frac{q^{-1}}{1-q^{-1/h}}\right)} \\
&= \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p\neq q,\ pq\le x^{1/h}}} \frac{1}{p(1-p^{-1/h}+p^{-1})} \frac{1}{q(1-q^{-1/h}+q^{-1})} - 2J_1 + J_2,
\end{aligned}
$$

where

$$
J_1 = \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p\neq q,\ pq\le x^{1/h}}} \frac{1}{p(1-p^{-1/h}+p^{-1})} \frac{q^{-\frac{1}{h}\left(\lfloor \frac{\log x}{\log q}\rfloor - h+1\right)}}{q(1-q^{-1/h}+q^{-1})}
$$

and

$$
J_2 = \gamma_{0,h} x^{1/h} \sum_{\substack{p,q \\ p\neq q,\ pq\le x^{1/h}}} \frac{p^{-\frac{1}{h}\left(\lfloor \frac{\log x}{\log p}\rfloor - h+1\right)}}{p(1-p^{-1/h}+p^{-1})} \frac{q^{-\frac{1}{h}\left(\lfloor \frac{\log x}{\log q}\rfloor - h+1\right)}}{q(1-q^{-1/h}+q^{-1})}.
$$

Using $\lfloor x\rfloor \ge x-1$, (2.1) and Lemma 2.3, we obtain

$$
J_1 \ll_h \sum_{\substack{p \\ p\le x^{1/h}/2}} \frac{1}{p(1-p^{-1/h}+p^{-1})} \left(\sum_{\substack{q \\ q\le x^{1/h}/p}} 1 \right) \ll_h \frac{x^{1/h}\log\log x}{\log x}.
$$

Similarly, we obtain

$$J_2 \ll_h \sum_{\substack{p \\ p\le x^{1/h}/2}} \frac{p^{-\frac{1}{h}\left(\left\lfloor\frac{\log x}{\log p}\right\rfloor - h+1\right)}}{p(1-p^{-1/h}+p^{-1})}\left(\sum_{\substack{q \\ q\le x^{1/h}/p}} 1\right) \ll_h \frac{\log\log x}{\log x}.$$

Combining the last three results, we obtain

$$\begin{aligned}\gamma_{0,h}x^{1/h} &\sum_{\substack{p,q \\ p\ne q,\ pq\le x^{1/h}}} \sum_{k=h}^{\left\lfloor\frac{\log x}{\log p}\right\rfloor}\sum_{l=h}^{\left\lfloor\frac{\log x}{\log q}\right\rfloor} \frac{1}{p^{k/h}\left(1+\frac{p^{-1}}{1-p^{-1/h}}\right)} \frac{1}{q^{l/h}\left(1+\frac{q^{-1}}{1-q^{-1/h}}\right)} \\ &= \gamma_{0,h}x^{1/h} \sum_{\substack{p,q \\ p\ne q,\ pq\le x^{1/h}}} \frac{1}{p(1-p^{-1/h}+p^{-1})}\frac{1}{q(1-q^{-1/h}+q^{-1})} \\ &\quad + O_h\left(\frac{x^{1/h}\log\log x}{\log x}\right).\end{aligned} \tag{4.25}$$

Thus, to complete the proof, we only require to estimate the main term in (4.25). Using Lemma 2.1, we have

$$\begin{aligned}&\sum_{\substack{p,q \\ p\ne q,\ pq\le x^{1/h}}} \frac{1}{p(1-p^{-1/h}+p^{-1})}\frac{1}{q(1-q^{-1/h}+q^{-1})} \\ &= \sum_{\substack{p,q \\ pq\le x^{1/h}}} \frac{1}{p(1-p^{-1/h}+p^{-1})}\frac{1}{q(1-q^{-1/h}+q^{-1})} - \sum_{p}\left(\frac{1}{p-p^{1-1/h}+1}\right)^2 \\ &\quad + O_h\left(\frac{1}{x^{1/(2h)}\log x}\right).\end{aligned} \tag{4.26}$$

Now, using

$$\frac{1}{p(1-p^{-1/h}+p^{-1})} = \frac{1}{p} + \frac{p^{-1/h}-p^{-1}}{p(1-p^{-1/h}+p^{-1})},$$

a similar result for another prime q and the symmetry of primes p and q, we write the first sum on the right-hand side of (4.26) as

$$\begin{aligned}&\sum_{\substack{p,q \\ pq\le x^{1/h}}} \frac{1}{p(1-p^{-1/h}+p^{-1})}\frac{1}{q(1-q^{-1/h}+q^{-1})} \\ &= \sum_{\substack{p,q \\ pq\le x^{1/h}}} \frac{1}{pq} + 2\sum_{\substack{p,q \\ pq\le x^{1/h}}} \frac{q^{-1/h}-q^{-1}}{pq(1-q^{-1/h}+q^{-1})} \\ &\quad + \sum_{\substack{p,q \\ pq\le x^{1/h}}} \frac{p^{-1/h}-p^{-1}}{p(1-p^{-1/h}+p^{-1})}\frac{q^{-1/h}-q^{-1}}{q(1-q^{-1/h}+q^{-1})}.\end{aligned}$$

The first sum on the right-hand side above is estimated using Lemma 2.4. For the second sum, we use Lemmas 2.1 and 2.3 and (3.3) to obtain

$$\sum_{\substack{p,q \\ pq \le x^{1/h}}} \frac{q^{-1/h} - q^{-1}}{pq(1 - q^{-1/h} + q^{-1})}$$

$$= \sum_{\substack{p \\ p \le x^{1/h}/2}} \frac{1}{p} \sum_{\substack{q \\ q \le x^{1/h}/p}} \frac{q^{-1/h} - q^{-1}}{q(1 - q^{-1/h} + q^{-1})}$$

$$= (\mathcal{L}_h(h+1) - \mathcal{L}_h(2h))(\log\log x + B_1 - \log h) + O_h\left(\frac{\log\log x}{\log x}\right),$$

and similarly, for the third sum, we obtain

$$\sum_{\substack{p,q \\ pq \le x^{1/h}}} \frac{p^{-1/h} - p^{-1}}{p(1 - p^{-1/h} + p^{-1})} \frac{q^{-1/h} - q^{-1}}{q(1 - q^{-1/h} + q^{-1})}$$

$$= (\mathcal{L}_h(h+1) - \mathcal{L}_h(2h))^2 + O_h\left(\frac{\log\log x}{x^{1/h^2}\log x}\right).$$

Combining the last three results with Lemma 2.4, we obtain

$$\sum_{\substack{p,q \\ pq \le x^{1/h}}} \frac{1}{p(1 - p^{-1/h} + p^{-1})} \frac{1}{q(1 - q^{-1/h} + q^{-1})}$$

$$= (\log\log x)^2 + 2D_1 \log\log x + D_1^2 - \zeta(2) + O_h\left(\frac{\log\log x}{\log x}\right).$$

Combining the above with (4.22)–(4.26) and using the first moment completes the proof for the second moment. □

5. Normal Order of $\omega(n)$ over h-Full Numbers

In this section, using the variance of $\omega(n)$ over the set of h-full numbers, we prove that $\omega(n)$ has normal order $\log\log n$ when restricted to this set.

Proof of Theorem 1.3. Notice that

$$\sum_{n \in \mathcal{N}_h(x)} (\omega(n) - \log\log n)^2$$

$$= \sum_{n \in \mathcal{N}_h(x)} \omega^2(n) - 2 \sum_{n \in \mathcal{N}_h(x)} \omega(n)\log\log n + \sum_{n \in \mathcal{N}_h(x)} (\log\log n)^2. \tag{5.1}$$

It is also well known that (see [8, Lemma 1])

$$|\mathcal{N}_h(x)| = \gamma_{0,h} x^{1/h} + O_h(x^{1/(h+1)}), \tag{5.2}$$

where $\gamma_{0,h}$ is defined in (1.4). Using the above with the first moment of $\omega(n)$ over h-full numbers given in Theorem 1.2 and the partial summation formula, we obtain

$$\sum_{n\in\mathcal{N}_h(x)} \omega(n)\log\log n = \gamma_{0,h}x^{1/h}(\log\log x)^2 + \gamma_{0,h}D_1x^{1/h}\log\log x + O_h\left(\frac{x^{1/h}\log\log x}{\log x}\right)$$

and

$$\sum_{n\in\mathcal{N}_h(x)} (\log\log n)^2 = \gamma_{0,h}x^{1/h}(\log\log x)^2 + O_h\left(\frac{x^{1/h}\log\log x}{\log x}\right).$$

Using the above two results and the second moment of $\omega(n)$ over the h-full numbers given in Theorem 1.2 in (5.1), we obtain

$$\sum_{n\in\mathcal{N}_h(x)} (\omega(n)-\log\log n)^2 = \gamma_{0,h}x^{1/h}\log\log x + D_2\gamma_{0,h}x^{1/h} + O_h\left(\frac{x^{1/h}\log\log x}{\log x}\right).$$

For the second part, let $g(x)$ be an increasing function such that $g(x)\to\infty$ as $x\to\infty$, and let E^h_{full} be the set of natural numbers $n\in\mathcal{N}_h$ with $x/\log x < n \le x$ such that

$$\frac{|\omega(n)-\log\log n|}{\sqrt{\log\log n}} \ge g(x).$$

Let $|E^h_{\text{full}}|$ be the cardinality of E^h_{full}. Then

$$\sum_{n\in\mathcal{N}_h(x)} (\omega(n)-\log\log n)^2 > g^2(x/\log x)|E^h_{\text{full}}|\log\log(x/\log x).$$

Using the asymptotic result for the left-hand side above, we deduce

$$\frac{|E^h_{\text{full}}|}{x^{1/h}} \ll_h \frac{\log\log x}{g^2(x/\log x)\log\log(x/\log x)}.$$

Since $g(x)\to\infty$ as $x\to\infty$, the right-hand side above goes to 0 as $x\to\infty$. Thus, $|E^h_{\text{full}}| = o(x^{1/h})$. Note that, by (5.2), $|\mathcal{N}_h(x)| \sim \gamma_{0,h}x^{1/h}$. Thus, the set of natural numbers $n\in\mathcal{N}_h$ with $x/\log x < n \le x$ such that

$$\frac{|\omega(n)-\log\log n|}{\sqrt{\log\log n}} \ge g(x),$$

is $o(|\mathcal{N}_h(x)|)$. Again, by (5.2), the remaining set of natural numbers $n\in\mathcal{N}_h(x/\log x)$ is $o(|\mathcal{N}_h(x)|)$. Together, this proves the first part of the theorem. Next, we choose ϵ' such that $0<\epsilon'<1/2$ and choose the function $g(x) = (\log\log x)^{\epsilon'}$. Let $\epsilon>0$ be arbitrary. Since $\epsilon'<1/2$, there exists $x_0\in\mathbb{R}$ such that for all $x\ge x_0$ and for all n

with $x/\log x < n \le x$,

$$\frac{(\log\log x)^{\epsilon'}}{(\log\log n)^{1/2}} < \frac{(\log\log x)^{\epsilon'}}{(\log\log(x/\log x))^{1/2}} \le \epsilon.$$

Hence, as $x \to \infty$, the set of numbers $n \in \mathcal{N}_h$ with $x/\log x \le n \le x$ that satisfy the inequality

$$\frac{|\omega(n) - \log\log n|}{\log\log n} \ge \epsilon,$$

is $o(|\mathcal{N}_h(x)|)$. This together with the fact that the remaining set of natural numbers $n \in \mathcal{N}_h(x/\log x)$ is $o(|\mathcal{N}_h(x)|)$ implies that $\omega(n)$ has normal order $\log\log n$ over h-full numbers. □

In this work, we employ a counting argument to establish that $\omega(n)$ has normal order $\log\log n$ over the h-free and h-full numbers. In addition, we can also establish that $\omega(n)$ satisfies the Gaussian distribution over the subsets of h-free and h-full numbers. We will report this result in a follow-up paper. The function field analog of this research has been studied by Lalín and Zhang [11].

Let $k \ge 1$ be a natural number. Let $\omega_k(n)$ denote the number of distinct prime factors of a natural number n with multiplicity k. Elma and Liu [3] studied the distribution of $\omega_k(n)$ over the natural numbers. They established that $\omega_1(n)$ behaves asymptotically similar to $\omega(n)$, whereas $\omega_k(n)$ with $k \ge 2$ is asymptotically smaller compared to $\omega(n)$ over the naturals. Moreover, over the natural numbers, $\omega_1(n)$ has normal order $\log\log n$, and $\omega_k(n)$ with $k \ge 2$ does not have a normal order. This naturally raises the question about the behavior of $\omega_k(n)$ over any subset of natural numbers, and whether they have the same normal order as $\omega(n)$ over such subsets. We will study the distribution of $\omega_k(n)$ over the particular sets of h-free and h-full numbers in a separate paper and answer the questions related to their behavior. Note that the function field analog of this research has been studied by Gómez and Lalín [5].

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