Journal of Number Theory 267 (2025) 176–201

Contents lists available at ScienceDirect

Journal of Number Theory

journal homepage: www.elsevier.com/locate/jnt

General Section

On the number of prime factors with a given multiplicity over h-free and h-full numbers ☆

Sourabhashis Das *, Wentang Kuo, Yu-Ru Liu

Department of Pure Mathematics, University of Waterloo, 200 University Avenue West, Waterloo, Ontario, N2L 3G1 Canada

A R T I C L E I N F O

Article history:
Received 29 February 2024
Received in revised form 22 August 2024
Accepted 26 August 2024
Available online 23 September 2024
Communicated by L. Smajlovic

MSC:
11N05
11N37

Keywords:
Prime factors
Prime-counting functions
Normal order
Erdős-Kac theorem

A B S T R A C T

Let k and n be natural numbers. Let $\omega_k(n)$ denote the number of distinct prime factors of n with multiplicity k as studied by Elma and the third author [5]. We obtain asymptotic estimates for the first and the second moments of $\omega_k(n)$ when restricted to the set of h-free and h-full numbers. We prove that $\omega_1(n)$ has normal order $\log\log n$ over h-free numbers, $\omega_h(n)$ has normal order $\log\log n$ over h-full numbers, and both of them satisfy the Erdős-Kac Theorem. Finally, we prove that the functions $\omega_k(n)$ with $1 < k < h$ do not have normal order over h-free numbers and $\omega_k(n)$ with $k > h$ do not have normal order over h-full numbers.

1. Introduction

For a natural number n, let the prime factorization of n be given as

$$n = p_1^{s_1} \cdots p_r^{s_r}, \tag{1}$$

☆ The research of W. Kuo and Y.-R. Liu are supported by NSERC grants RGPIN-2020-03915 and 50503-11689 respectively.

* Corresponding author.
E-mail addresses: souravasis@gmail.com, s57das@uwaterloo.ca (S. Das).

https://doi.org/10.1016/j.jnt.2024.08.007

where p_i's are its distinct prime factors and s_i's are their respective multiplicities. Let $\omega(n)$ denote the total number of distinct prime factors in the factorization of n. Thus, $\omega(n) = r$. The average distribution of $\omega(n)$ over natural numbers is well-known (see [13, Theorem 430]):

$$\sum_{n\le x} \omega(n) = x \log\log x + B_1 x + O\left(\frac{x}{\log x}\right), \tag{2}$$

where B_1 is the Mertens constant given by

$$B_1 = \gamma + \sum_p \left(\log\left(1 - \frac{1}{p}\right) + \frac{1}{p}\right), \tag{3}$$

with $\gamma \approx 0.57722$, the Euler-Mascheroni constant, and where the sum runs over all primes p.

Let $h \geq 2$ be an integer. Let n be a natural number with the factorization given in (1). We say n is *h-free* if $s_i \leq h-1$ for all $i \in \{1, \cdots, r\}$, and we say n is *h-full* if $s_i \geq h$ for all $i \in \{1, \cdots, r\}$. Let $\mathcal{S}_h$ denote the set of h-free numbers and let $\mathcal{N}_h$ denote the set of all h-full numbers. Let $\gamma_{0,h}$ be the constant defined as

$$\gamma_{0,h} := \prod_p \left(1 + \frac{p - p^{1/h}}{p^2(p^{1/h} - 1)}\right), \tag{4}$$

where the product runs over all primes p, and let $\mathcal{L}_h(r)$ be the convergent sum defined for $r > h$ as

$$\mathcal{L}_h(r) := \sum_p \frac{1}{p^{(r/h)-1}\left(p - p^{1-1/h} + 1\right)}. \tag{5}$$

In [3, Theorem 1.1 and Theorem 1.2], the authors proved the following distribution results for $\omega(n)$ restricted to the sets of h-free numbers and h-full numbers:

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}} \omega(n) = \frac{1}{\zeta(h)} x \log\log x + \left(B_1 - \sum_p \frac{p-1}{p(p^h - 1)}\right)\frac{x}{\zeta(h)} + O_h\left(\frac{x}{\log x}\right), \tag{6}$$

and

$$\sum_{\substack{n\le x\\ n\in\mathcal{N}_h}} \omega(n) = \gamma_{0,h} x^{1/h} \log\log x + \left(B_1 - \log h + \mathcal{L}_h(h+1) - \mathcal{L}_h(2h)\right)\gamma_{0,h} x^{1/h} + O_h\left(\frac{x^{1/h}}{\log x}\right), \tag{7}$$

where $\zeta(s)$ represents the classical Riemann ζ-function, and where O_h denotes big-O with the implied constant depending on h.

Given natural numbers n and k, let $\omega_k(n)$ denote the number of distinct prime factors of n with multiplicity k. Note that

$$\omega(n) = \sum_{k \geq 1} \omega_k(n).$$

Let $P(k)$ denote the convergent sum given by

$$P(k) := \sum_p \frac{1}{p^k}. \tag{8}$$

Using this definition, the distributions of average value of $\omega_k(n)$ over natural numbers were proved by Elma and the third author [5, Theorem 1.1] as

$$\sum_{n \leq x} \omega_1(n) = x \log\log x + (B_1 - P(2))\, x + O\left(\frac{x}{\log x}\right), \tag{9}$$

and for $k \geq 2$,

$$\sum_{n \leq x} \omega_k(n) = (P(k) - P(k+1))\, x + O\left(x^{\frac{k+1}{3k-1}} (\log x)^2\right). \tag{10}$$

The results in (2), (9) and (10) suggest that $\omega(n)$ and $\omega_1(n)$ share a similar asymptotic distribution with only a difference in the coefficient of the second main term, whereas the average distribution of $\omega_k(n)$ with $k \geq 2$ is smaller than that of $\omega(n)$. We verify this in our work as well. In the next two theorems, we show that when restricted to the set of h-free numbers, $\omega_1(n)$ behaves similarly to $\omega(n)$ and $\omega_k(n)$ with $k \geq 2$ exhibits a smaller asymptotic size that $\omega(n)$. We begin by studying the distributions of $\omega_1(n)$ over h-free numbers. We define the constants

$$C_1 := B_1 - \sum_p \frac{p^{h-1} - 1}{p(p^h - 1)}, \tag{11}$$

and

$$C_2 := C_1^2 + C_1 - \zeta(2) - \sum_p \left(\frac{p^{h-1} - p^{h-2}}{p^h - 1}\right)^2.$$

For the first and the second moment of $\omega_1(n)$ over h-free numbers, we prove:

Theorem 1.1. *Let $x > 2$ be a real number. Let $h \geq 2$ be an integer. Let $\mathcal{S}_h$ be the set of h-free numbers. Then, we have*

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}} \omega_1(n) = \frac{1}{\zeta(h)} x\log\log x + \frac{C_1}{\zeta(h)} x + O_h\left(\frac{x}{\log x}\right),$$

and

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}} \omega_1^2(n) = \frac{1}{\zeta(h)} x(\log\log x)^2 + \frac{2C_1+1}{\zeta(h)} x\log\log x + \frac{C_2}{\zeta(h)} x + O_h\left(\frac{x}{\log x}\right).$$

Next, we establish the moments of $\omega_k(n)$ with $k \ge 2$ over h-free numbers as the following:

Theorem 1.2. *Let $k \ge 2$ and $h \ge 2$ be any integers. Let $\mathcal{S}_h$ be the set of h-free numbers. For $k \le (h-1)$, we have*

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}} \omega_k(n) = \sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)}\right) \frac{x}{\zeta(h)} + O_{h,k}\left(\frac{x^{1/k}}{\log x}\right),$$

and

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}} \omega_k^2(n)$$
$$= \left(\left(\sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)}\right)\right)^2 - \sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)}\right)^2 + \sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)}\right)\right) \frac{x}{\zeta(h)}$$
$$+ O_{h,k}\left(\frac{x^{1/k}\log\log x}{\log x}\right),$$

where $O_{h,k}$ means that the implied constant depends on h and k.

Remark 1.1. Note that if $n \in \mathcal{S}_h$, then $\omega_k(n) = 0$ for all $k \ge h$. Thus, the distribution of $\omega_k(n)$ with $k \ge h$ over the h-free numbers is zero.

Next, we prove the distribution of $\omega_k(n)$ over h-full numbers. We notice that $\omega(n)$ and $\omega_h(n)$ have similar asymptotic distributions over h-full numbers. Moreover, $\omega_k(n)$ with $k > h$ has a smaller asymptotic size than $\omega(n)$. We can thus infer that the smallest power of primes defining the subset of h-full numbers contributes to the main term for the asymptotic distribution of $\omega(n)$ over the subset. This inference also satisfies the behavior observed over h-free numbers. The set of h-free numbers includes the first power of primes, and it is observed that $\omega(n)$ and $\omega_1(n)$, which counts prime factors with multiplicity 1, satisfy similar distributions over h-free numbers. To prove the distribution over h-full numbers, we define two new constants

$$D_1 := B_1 - \log h - \mathcal{L}_h(2h), \tag{12}$$

and

$$D_2 := D_1^2 + D_1 - \zeta(2) - \sum_p \left(\frac{p^{1/h} - 1}{p^{1+1/h} - p + p^{1/h}} \right)^2, \tag{13}$$

where $\mathcal{L}_h(\cdot)$ is defined in (5). For the first moments of $\omega_k(n)$ over h-full numbers, we prove:

Theorem 1.3. *Let $k \geq 2$ and $h \geq 2$ be any integers. Let $\mathcal{N}_h$ be the set of h-full numbers. Then, we have*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_h(n) = \gamma_{0,h} x^{1/h} \log\log x + D_1 \gamma_{0,h} x^{1/h} + O_h\left(\frac{x^{1/h}}{\log x} \right),$$

where $\gamma_{0,h}$ is defined in (4).

Moreover, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_{h+1}(n) = \left(\mathcal{L}_h(h+1) - \mathcal{L}_h(h+2)\right) \gamma_{0,h} x^{1/h} + O_h\left(x^{1/(h+1)} \log\log x \right),$$

and for $k > h+1$, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_k(n) = \left(\mathcal{L}_h(k) - \mathcal{L}_h(k+1)\right) \gamma_{0,h} x^{1/h} + O_{h,k}\left(x^{1/(h+1)} \right).$$

For the second moments, we obtain:

Theorem 1.4. *Under the assumptions as in Theorem 1.3, we have*

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_h^2(n) = \gamma_{0,h} x^{1/h} (\log\log x)^2 + (2D_1 + 1)\, \gamma_{0,h} x^{1/h} \log\log x + D_2 \gamma_{0,h} x^{1/h} + O_h\left(\frac{x^{1/h} \log\log x}{\log x} \right).$$

Moreover, for $k = h+1$, we have,

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_{h+1}^2(n) = \Bigg(\left(\mathcal{L}_h(h+1) - \mathcal{L}_h(h+2)\right)^2 + \mathcal{L}_h(h+1) - \mathcal{L}_h(h+2)$$

$$- \sum_{p} \left(\frac{p^{1/h} - 1}{p^{1+2/h} - p^{1+1/h} + p^{2/h}} \right)^2 \Bigg) \gamma_{0,h} x^{1/h} + O_h \left(x^{1/(h+1)} (\log\log x)^2 \right),$$

and for $k > h+1$, we have

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}_h}} \omega_k^2(n)$$

$$= \Bigg(\left(\mathcal{L}_h(k) - \mathcal{L}_h(k+1)\right)^2 + \mathcal{L}_h(k) - \mathcal{L}_h(k+1)$$

$$- \sum_{p} \left(\frac{p^{1/h} - 1}{p^{(k+1)/h} - p^{k/h} + p^{(k+1-h)/h}} \right)^2 \Bigg) \gamma_{0,h} x^{1/h} + O_{h,k} \left(x^{1/(h+1)} \right).$$

Remark 1.2. Note that if $n \in \mathcal{N}_h$, then $\omega_k(n) = 0$ for all $k \le (h-1)$. Thus, the distribution of $\omega_k(n)$ over h-full numbers is zero for $k \le (h-1)$.

We recall the definition of normal order over a subset of natural numbers as mentioned in [3]. Let $S \subseteq \mathbb{N}$ and $S(x)$ denote the set of natural numbers belonging to S and less than or equal to x. Let $f, F : S \to \mathbb{R}_{\ge 0}$ be two functions such that F is non-decreasing. Then, $f(n)$ is said to have normal order $F(n)$ over S if for any $\epsilon > 0$, the number of $n \in S(x)$ that do not satisfy the inequality

$$(1-\epsilon)F(n) \le f(n) \le (1+\epsilon)F(n)$$

is $o(S(x))$ as $x \to \infty$. Hardy and Ramanujan [12] proved that $\omega(n)$ has the normal order $\log\log n$ over naturals. In fact, the authors in [3] showed that $\omega(n)$ has the normal order $\log\log n$ over h-free and over h-full numbers as well. Note that the set of h-free numbers has positive density, and both ω and ω_1 behave asymptotically similar over h-free numbers. Thus, the proof of $\omega_1(n)$ having normal order $\log\log n$ over h-free numbers follows from the classical case. In particular, one can establish that for any $\epsilon > 0$, the number of $n \in \mathcal{S}_h(x)$ that do not satisfy the inequality

$$(1-\epsilon)\log\log n \le \omega_1(n) \le (1+\epsilon)\log\log n$$

is $o(|\mathcal{S}_h(x)|)$ as $x \to \infty$. On the other hand, the set of h-full numbers has density 0, and thus the proof of normal order of ω_h does not follow from the classical result. However, since ω and ω_h behave asymptotically similar over h-full numbers, the proof of $\omega_h(n)$ having normal order $\log\log n$ over h-full numbers can be inferred in a manner analogous to the proof presented in [3, Theorem 1.3] for $\omega(n)$. In particular, one can establish that for any $\epsilon > 0$, the number of $n \in \mathcal{N}_h(x)$ that do not satisfy the inequality

$$(1-\epsilon)\log\log n \le \omega_h(n) \le (1+\epsilon)\log\log n$$

is $o(|\mathcal{N}_h(x)|)$ as $x \to \infty$.

In [6], Erdős and Kac established a pioneering result that $\omega(n)$ obeys the Gaussian distribution over naturals. In particular, they proved

$$\lim_{x\to\infty} \frac{1}{x}\left|\left\{n \leq x \ : \ \frac{\omega(n) - \log\log n}{\sqrt{\log\log n}} \leq a\right\}\right| = \Phi(a),$$

where

$$\Phi(a) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{a} e^{-u^2/2}\, du. \tag{14}$$

Following their work, various approaches to the Erdős-Kac theorem have been pursued, for example, see [2,7–11,15]. In [4], the authors showed that $\omega(n)$ satisfies the Erdős-Kac theorem over the subsets of h-free and h-full ideals of a number field, thus in particular, over h-free and h-full numbers. We extend this result to $\omega_1(n)$ over h-free numbers and $\omega_h(n)$ over h-full numbers. We prove the following two results:

Theorem 1.5. *Let $x > 2$ be any real number and $h \geq 2$ be any integer. Let $\mathcal{S}_h(x)$ denote the set of h-free numbers less than or equal to x. Then for $a \in \mathbb{R}$, we have*

$$\lim_{x\to\infty} \frac{1}{|\mathcal{S}_h(x)|}\left|\left\{n \in \mathcal{S}_h(x) \ : \ \frac{\omega_1(n) - \log\log n}{\sqrt{\log\log n}} \leq a\right\}\right| = \Phi(a),$$

where $\Phi(a)$ is defined in (14).

Theorem 1.6. *Let $x > 2$ be any real number and $h \geq 2$ be any integer. Let $\mathcal{N}_h(x)$ denote the set of h-free numbers less than or equal to x. Then for $a \in \mathbb{R}$, we have*

$$\lim_{x\to\infty} \frac{1}{|\mathcal{N}_h(x)|}\left|\left\{n \in \mathcal{N}_h(x) \ : \ \frac{\omega_h(n) - \log\log n}{\sqrt{\log\log n}} \leq a\right\}\right| = \Phi(a),$$

where $\Phi(a)$ is defined in (14).

Unlike $\omega_1(n)$, we observe that $\omega_k(n)$ for $1 < k < h$ does not have a normal order. This goes in accordance with the findings of Elma and the third author [5] where they proved that $\omega_k(n)$ for $k > 1$ does not have a normal order over natural numbers. In particular, we prove:

Theorem 1.7. *For any integer $h \geq 2$ and any integer k satisfying $1 < k < h$, the function $\omega_k(n)$ does not have normal order $F(n)$ for any non-decreasing function $F : \mathcal{S}_h \to \mathbb{R}_{\geq 0}$.*

Finally, we show that $\omega_k(n)$ for $k > h$ does not have a normal order over h-full numbers. In particular, we prove:

Theorem 1.8. *For any integer $h \geq 2$ and any integer $k > h$, the function $\omega_k(n)$ does not have normal order $F(n)$ for any non-decreasing function $F : \mathcal{N}_h \to \mathbb{R}_{\geq 0}$.*

For a natural number n, let $\Omega(n)$ denote the number of prime factors of n counted with multiplicity. In particular, for the representation of n given in (1), $\Omega(n) = \sum_{i=1}^{r} s_i$. Let $\Omega_k(n)$ be the number of prime factors of f with a given multiplicity $k \geqslant 1$. Note that $\Omega_k(n) = k \cdot \omega_k(n)$ and $\Omega(n) = \sum_{k \geqslant 1} \Omega_k(n)$ for all $n \in \mathbb{N}$. One can deduce similar results for $\Omega_k(n)$ as our results in this paper. In particular, one can prove that $\Omega_1(n)$ has normal order $\log\log n$ over h-free numbers, $\Omega_h(n)$ has normal order $h \log\log n$ over h-full numbers, and both satisfy the Erdős-Kac Theorem. One can also prove that the functions $\Omega_k(n)$ with $1 < k < h$ do not have normal order over h-free numbers and $\Omega_k(n)$ with $k > h$ do not have normal order over h-full numbers.

2. Lemmata

First, we recall the following results necessary for our study:

Lemma 2.1. *[1, Lemma 1.2] If $k > 1$ be any real number. Then*

$$\sum_{p \geq x} \frac{1}{p^k} = \frac{1}{(k-1)x^{k-1}(\log x)} + O\left(\frac{1}{x^{k-1}(\log x)^2}\right).$$

Lemma 2.2. *[3, Lemma 2.2] Let $\alpha > 0$ be any real number satisfying $0 < \alpha < 1$. Then*

$$\sum_{p \leq x} \frac{1}{p^\alpha} = O_\alpha\left(\frac{x^{1-\alpha}}{\log x}\right).$$

Lemma 2.3. *Let $h \geq 2$ be a fixed integer. Let $y > 2$ and $r > h$ be fixed real numbers. Then*

$$\sum_{p \leq y} \frac{1}{p^{r/h}\left(1 - p^{-1/h} + p^{-1}\right)} = \mathcal{L}_h(r) + O_{h,r}\left(\frac{1}{y^{\frac{r}{h}-1}(\log y)}\right),$$

where $\mathcal{L}_h(r)$ is the convergent sum defined in (5) as

$$\mathcal{L}_h(r) := \sum_{p} \frac{1}{p^{(r/h)-1}\left(p - p^{1-1/h} + 1\right)},$$

and $O_{h,r}$ means that the implied constant depends on both h and r.

Proof. Note that

$$\sum_{p \leq y} \frac{1}{p^{r/h}\left(1 - p^{-1/h} + p^{-1}\right)} = \mathcal{L}_h(r) + O_h\left(\sum_{p > y} \frac{1}{p^{r/h}}\right).$$

Applying Lemma 2.1 with $k = r/h$ completes the proof. □

Lemma 2.4. *[16, Exercise 9.4.4] For $x > 2$, we have*

$$\sum_{p \le x/2} \frac{1}{p \log(x/p)} = O\left(\frac{\log\log x}{\log x}\right).$$

Lemma 2.5. *[3, Lemma 2.4] Let p and q denote prime numbers. For $x > 2$, we have*

$$\sum_{\substack{p,q \\ pq \le x}} \frac{1}{pq} = (\log\log x)^2 + 2B_1 \log\log x + B_1^2 - \zeta(2) + O\left(\frac{\log\log x}{\log x}\right).$$

Next, we recall the following results regarding the density of certain sequences of h-free and h-full numbers:

Lemma 2.6. *[14, Lemma 3] Let $x > 2$ be a real number. Let $h \ge 2$ be an integer. Let $\mathcal{S}_h$ be the set of h-free numbers. Let $q_1, \cdots, q_r$ be prime numbers. Then, we have*

$$\sum_{\substack{n \le x, n \in \mathcal{S}_h \\ (n,q_1) = \cdots = (n,q_r) = 1}} 1 = \prod_{i=1}^{r} \left(\frac{q_i^h - q_i^{h-1}}{q_i^h - 1}\right) \frac{x}{\zeta(h)} + O_h\left(2^r x^{1/h}\right).$$

Let $C_{r,h}$ be a constant dependent on r and h, defined as

$$C_{r,h} := \prod_{j=h, j \ne r}^{2h-1} \zeta(j/r),$$

and let $\phi_h(s)$ be a complex valued function defined on $\Re(s) > 1/(2h+3)$, satisfying the equation

$$\prod_p \left(1 - p^{-(2h+2)s} + \sum_{r=2h+3}^{(3h^2+h-2)/2} a_{r,h} p^{-rs}\right) = \zeta^{-1}((2h+2)s)\phi_h(s),$$

where $a_{r,h}$ satisfying the identity

$$\left(1 + \frac{v^h}{1-v}\right)(1 - v^h) \cdots (1 - v^{2h-1}) = 1 - v^{2h+2} + \sum_{2h+3}^{(3h^2+h-2)/2} a_{r,h} v^r.$$

Lemma 2.7. *[3, Lemma 4.1] Let $q_1, q_2, \cdots, q_r$ be distinct primes. Let*

$$A_{q_1,\cdots,q_r,h}(x) := \sum_{\substack{n \le x, n \in \mathcal{N}_h \\ (n,q_1) = \cdots = (n,q_r) = 1}} 1. \tag{15}$$

For any $x > 2$, we have

$$A_{q_1,\cdots,q_r,h}(x) = \gamma_{q_1,\cdots,q_r,0,h}x^{\frac{1}{h}} + \gamma_{q_1,\cdots,q_r,1,h}x^{\frac{1}{h+1}} + \cdots + \gamma_{q_1,\cdots,q_r,h-1,h}x^{\frac{1}{2h-1}} + O_h\left(x^{\eta_h}\right),$$

where $\frac{1}{2h+2} < \eta_h < \frac{1}{2h-1}$, and for $i \in \{0, 1, \cdots, h-1\}$,

$$\gamma_{q_1,\cdots,q_r,i,h} = C_{h+i,h}\frac{\phi_h(1/(h+i))}{\zeta((2h+2)/(h+i))\left(\prod_{j=1}^{r}\left(1 + \frac{q_j^{-h/(h+i)}}{1-q_j^{-1/(h+i)}}\right)\right)}.$$

Lemma 2.8. *[14, Lemma 17] Let q be a prime and $k > h$ be integers. Then, for any small $\epsilon > 0$, we have*

$$\sum_{\substack{n\le x \\ n\in\mathcal{N}_h\cap\mathcal{S}_k,(n,q)=1}} 1 = \frac{1-q^{-1/h}}{1-q^{-1/h}+q^{-1}-q^{-k/h}}\eta_{h,k}x^{1/h} + O\left(x^{\frac{2h+1}{2h(h+1)}+\epsilon}\right),$$

where $\eta_{h,k}$ is the convergent product given by

$$\eta_{h,k} = \prod_p\left(1-\frac{1}{p}\right)\left(\frac{1-p^{-1/h}+p^{-1}-p^{-k/h}}{1-p^{-1/h}}\right). \tag{16}$$

3. Distribution of $\omega_k(n)$ over h-free numbers

In this section, we study the distribution of the average value of $\omega_k(n)$ over h-free numbers.

3.1. For $\omega_1(n)$

Proof of Theorem 1.1. Writing $n = py$ with $(y, p) = 1$, we obtain

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}}\omega_1(n) = \sum_{\substack{n\le x\\ n\in\mathcal{S}_h}}\sum_{\substack{p\\ p\|n}}1 = \sum_{p\le x}\sum_{\substack{n\le x\\ n\in\mathcal{S}_h,p\|n}}1 = \sum_{p\le x}\sum_{\substack{y\le x/p\\ y\in\mathcal{S}_h,(y,p)=1}}1.$$

Now, first using Lemma 2.6 for a single prime p to the above and then using Lemma 2.2, we obtain

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}}\omega_1(n) = \sum_{p\le x}\left(\frac{p^h-p^{h-1}}{p(p^h-1)}\right)\frac{x}{\zeta(h)} + O_h\left(\frac{x}{\log x}\right).$$

Using Lemma 2.1, $\frac{p^h-p^{h-1}}{p(p^h-1)} = \frac{1}{p} - \frac{p^{h-1}-1}{p(p^h-1)}$, and Mertens' second theorem given by

$$\sum_{p\le x}\frac{1}{p}=\log\log x+B_1+O\left(\frac{1}{\log x}\right), \tag{17}$$

we obtain

$$\sum_{p\le x}\left(\frac{p^h-p^{h-1}}{p(p^h-1)}\right)=\log\log x+B_1-\sum_{p}\frac{p^{h-1}-1}{p(p^h-1)}+O_h\left(\frac{1}{\log x}\right). \tag{18}$$

Plugging the above back into the previous equation completes the first part of the proof.

Next, note that

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}}\omega_1^2(n)=\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}}\left(\sum_{\substack{p\\ p\|n}}1\right)^2=\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}}\omega_1(n)+\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}}\sum_{\substack{p,q\\ p\|n,q\|n,p\neq q}}1, \tag{19}$$

where p and q above denote primes. The first sum on the right-hand side above is the first moment estimated above and for the second sum, using Lemma 2.6 for two primes p and q, we obtain

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}}\sum_{\substack{p,q\\ p\|n,q\|n,p\neq q}}1=\sum_{\substack{p,q\\ p\neq q,pq\le x}}\left(\left(\frac{p^h-p^{h-1}}{p(p^h-1)}\right)\left(\frac{q^h-q^{h-1}}{q(q^h-1)}\right)\frac{x}{\zeta(h)}+O_h\left(\frac{x^{1/h}}{(pq)^{1/h}}\right)\right). \tag{20}$$

Next, we bound the above error term using Lemma 2.2 with $\alpha=1/h$ and Lemma 2.4 as the following

$$\begin{aligned} x^{1/h}\sum_{\substack{p,q\\ p\neq q,pq\le x}}\frac{1}{(pq)^{1/h}}&=x^{1/h}\sum_{p\le x/2}\frac{1}{p^{1/h}}\sum_{q\le x/p}\frac{1}{q^{1/h}}\\ &\ll_h x\sum_{p\le x/2}\left(\frac{1}{p\log(x/p)}\right)\\ &\ll_h \frac{x\log\log x}{\log x}. \end{aligned} \tag{21}$$

Now, we estimate the main term in (20). First, we can divide the sum as

$$\begin{aligned} &\sum_{\substack{p,q\\ p\neq q,pq\le x}}\left(\frac{p^h-p^{h-1}}{p(p^h-1)}\right)\left(\frac{q^h-q^{h-1}}{q(q^h-1)}\right)\\ &=\sum_{\substack{p,q\\ pq\le x}}\left(\frac{p^h-p^{h-1}}{p(p^h-1)}\right)\left(\frac{q^h-q^{h-1}}{q(q^h-1)}\right)-\sum_{\substack{p\\ p\le x^{1/2}}}\left(\frac{p^h-p^{h-1}}{p(p^h-1)}\right)^2. \end{aligned} \tag{22}$$

The second sum on the right-hand side above is estimated using Lemma 2.1 as

$$\sum_{\substack{p \\ p\le x^{1/2}}} \left(\frac{p^h - p^{h-1}}{p(p^h-1)}\right)^2 = \sum_p \left(\frac{p^h - p^{h-1}}{p(p^h-1)}\right)^2 + O\left(\frac{1}{x^{1/2}\log x}\right). \tag{23}$$

Using $\frac{p^h-p^{h-1}}{p(p^h-1)} = \frac{1}{p} - \frac{p^{h-1}-1}{p(p^h-1)}$, and the symmetry in p and q, we have

$$\begin{aligned}
&\sum_{\substack{p,q \\ pq\le x}} \left(\frac{p^h - p^{h-1}}{p(p^h-1)}\right)\left(\frac{q^h - q^{h-1}}{q(q^h-1)}\right) \\
&= \sum_{\substack{p,q \\ pq\le x}} \frac{1}{pq} - 2\sum_{\substack{p,q \\ pq\le x}} \frac{1}{p}\left(\frac{q^{h-1}-1}{q(q^h-1)}\right) + \sum_{\substack{p,q \\ pq\le x}} \left(\frac{p^{h-1}-1}{p(p^h-1)}\right)\left(\frac{q^{h-1}-1}{q(q^h-1)}\right). \qquad (24)
\end{aligned}$$

We estimate the sums on the right-hand side above separately. For the first sum, we use Lemma 2.5. For the second sum, we use Lemma 2.1, and then (17) and the classical prime number theorem given as

$$\sum_{p\le x} 1 = \frac{x}{\log x} + \left(\frac{x}{(\log x)^2}\right) \tag{25}$$

to obtain

$$\begin{aligned}
&\sum_{\substack{p,q \\ pq\le x}} \frac{1}{p}\left(\frac{q^{h-1}-1}{q(q^h-1)}\right) \\
&= \sum_{\substack{p \\ p\le x/2}} \frac{1}{p}\left(\sum_p \left(\frac{p^{h-1}-1}{p(p^h-1)}\right) + O\left(\frac{1}{(x/p)\log(x/p)}\right)\right) \\
&= \sum_p \left(\frac{p^{h-1}-1}{p(p^h-1)}\right)(\log\log x + B_1) + O\left(\frac{1}{\log x}\right). \qquad (26)
\end{aligned}$$

For the third sum, we use Lemma 2.1 twice and then Lemma 2.4 to obtain

$$\sum_{\substack{p,q \\ pq\le x}} \left(\frac{p^{h-1}-1}{p(p^h-1)}\right)\left(\frac{q^{h-1}-1}{q(q^h-1)}\right) = \left(\sum_p \left(\frac{p^{h-1}-1}{p(p^h-1)}\right)\right)^2 + O\left(\frac{\log\log x}{x\log x}\right). \tag{27}$$

Combining (24), (26), (27), and Lemma 2.5, we obtain

$$\begin{aligned}
&\sum_{\substack{p,q \\ pq\le x}} \left(\frac{p^h - p^{h-1}}{p(p^h-1)}\right)\left(\frac{q^h - q^{h-1}}{q(q^h-1)}\right) \\
&= (\log\log x)^2 + 2B_1\log\log x + B_1^2 - \zeta(2) - 2\sum_p \left(\frac{p^{h-1}-1}{p(p^h-1)}\right)(\log\log x + B_1)
\end{aligned}$$

$$+\left(\sum_{p}\frac{p^{h-1}-1}{p(p^h-1)}\right)^2+O\left(\frac{\log\log x}{\log x}\right).$$

Combining (19), (20), (21), (22) and (23), and then combining it further with the above equation and the first moment of $\omega_1(n)$ over h-free numbers, we obtain the second moment estimate. □

3.2. For $\omega_k(n)$ with $k\geq 2$

Proof of Theorem 1.2. Using $n=p^k y$ with $(p,y)=1$, we obtain

$$\sum_{\substack{n\leq x\\ n\in\mathcal{S}_h}}\omega_k(n)=\sum_{\substack{n\leq x\\ n\in\mathcal{S}_h}}\sum_{\substack{p|n\\ p^k\|n}}1=\sum_{p\leq x^{1/k}}\sum_{\substack{n\leq x\\ n\in\mathcal{S}_h,p^k\|n}}1=\sum_{p\leq x^{1/k}}\sum_{\substack{y\leq x/p^k\\ y\in\mathcal{S}_h,(p,y)=1}}1. \tag{28}$$

Using Lemma 2.6 for a single prime p, we obtain

$$\begin{aligned}\sum_{p\leq x^{1/k}}\sum_{\substack{y\leq x/p^k\\ y\in\mathcal{S}_h,(p,y)=1}}1&=\sum_{p\leq x^{1/k}}\left(\frac{1}{\zeta(h)}\left(\frac{p^h-p^{h-1}}{p^h-1}\right)\frac{x}{p^k}+O_h\left(\frac{x^{1/h}}{p^{k/h}}\right)\right)\\&=\sum_{p}\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)\frac{x}{\zeta(h)}+\sum_{p>x^{1/k}}\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)\frac{x}{\zeta(h)}\\&\quad+O_h\left(x^{1/h}\sum_{p\leq x^{1/k}}\frac{1}{p^{k/h}}\right).\end{aligned} \tag{29}$$

Using Lemma 2.1, we estimate the second sum in the above equation as

$$\sum_{p>x^{1/k}}\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)\frac{x}{\zeta(h)}\ll_{h,k}\frac{x^{1/k}}{\log x}. \tag{30}$$

Using Lemma 2.2 with $\alpha=k/h$, we obtain

$$\sum_{p\leq x^{1/k}}\frac{1}{p^{k/h}}\ll_{h,k}\frac{x^{\frac{1}{k}-\frac{1}{h}}}{\log x}. \tag{31}$$

Combining (28), (29), (30), and (31) completes the first part of the proof.

For the second moment, note that

$$\sum_{\substack{n\leq x\\ n\in\mathcal{S}_h}}\omega_k^2(n)=\sum_{\substack{n\leq x\\ n\in\mathcal{S}_h}}\left(\sum_{\substack{p\\ p^k\|n}}1\right)^2=\sum_{\substack{n\leq x\\ n\in\mathcal{S}_h}}\omega_k(n)+\sum_{\substack{n\leq x\\ n\in\mathcal{S}_h}}\sum_{\substack{p,q\\ p^k\|n,q^k\|n,p\neq q}}1, \tag{32}$$

where p and q above denote primes. The first sum on the right-hand side above is the first moment studied above, and for the second sum, using Lemma 2.6, we obtain

$$\sum_{\substack{n\le x\\ n\in\mathcal{S}_h}}\ \sum_{\substack{p,q\\ p^k\|n,q^k\|n,p\neq q}} 1 = \sum_{\substack{p,q\\ p\neq q,pq\le x^{1/k}}} \left(\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)\left(\frac{q^h-q^{h-1}}{q^k(q^h-1)}\right)\frac{x}{\zeta(h)} + O_h\left(\frac{x^{1/h}}{(pq)^{k/h}}\right)\right). \tag{33}$$

Next, we bound the above error term. We employ Lemma 2.2 with $\alpha = k/h$, and with $\alpha = 2k/h$ when $2k < h$, (17) when $2k = h$, and $\sum_p 1/p^{2k/h} = O(1)$ when $2k > h$ below to obtain

$$x^{1/h}\sum_{\substack{p,q\\ p\neq q,pq\le x^{1/k}}}\frac{1}{(pq)^{k/h}} = x^{1/h}\sum_{\substack{p,q\\ pq\le x^{1/k}}}\frac{1}{(pq)^{k/h}} - x^{1/h}\sum_{p\le x^{1/2k}}\frac{1}{p^{2k/h}} \ll_{k,h} \frac{x^{\frac{1}{k}}\log\log x}{\log x}. \tag{34}$$

Now, we estimate the main term in (33). First, we can divide the sum as

$$\sum_{\substack{p,q\\ p\neq q,pq\le x^{1/k}}}\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)\left(\frac{q^h-q^{h-1}}{q^k(q^h-1)}\right) = \sum_{\substack{p,q\\ pq\le x^{1/k}}}\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)\left(\frac{q^h-q^{h-1}}{q^k(q^h-1)}\right) - \sum_{\substack{p\\ p\le x^{1/2k}}}\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)^2. \tag{35}$$

The second sum on the right-hand side above is estimated using Lemma 2.1 as

$$\sum_{\substack{p\\ p\le x^{1/2k}}}\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)^2 = \sum_p\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)^2 + O_k\left(\frac{1}{x^{1-\frac{1}{2k}}\log x}\right). \tag{36}$$

For the first sum, we employ Lemma 2.1, then Lemma 2.4, and then again Lemma 2.1 to obtain

$$\sum_{\substack{p,q\\ pq\le x^{1/k}}}\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)\left(\frac{q^h-q^{h-1}}{q^k(q^h-1)}\right) = \sum_{\substack{p\\ p\le x^{1/k}/2}}\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)\left(\sum_p\left(\frac{p^h-p^{h-1}}{p^k(p^h-1)}\right)\right) + O_k\left(x^{\frac{1}{k}-1}\sum_{\substack{p\\ p\le x^{1/k}/2}}\frac{1}{p\log(x^{1/k}/p)}\right)$$

$$= \left(\sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)} \right) \right)^2 + O_k \left(\frac{x^{\frac{1}{k}-1}}{\log x} \right) + O_k \left(\frac{x^{\frac{1}{k}-1} \log\log x}{\log x} \right).$$

Combining the last three results, we obtain

$$\sum_{\substack{p,q \\ p \neq q, pq \leq x^{1/k}}} \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)} \right) \left(\frac{q^h - q^{h-1}}{q^k(q^h-1)} \right) \frac{x}{\zeta(h)}$$

$$= \left(\sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)} \right) \right)^2 - \sum_p \left(\frac{p^h - p^{h-1}}{p^k(p^h-1)} \right)^2 + O_{h,k} \left(\frac{x^{1/k} \log\log x}{\log x} \right).$$

Combining the above with (32), (33), and the first moment of $\omega_k(n)$ studied in the first part of the proof, we obtain the required result. □

4. Distribution of $\omega_k(n)$ over h-full numbers

In this section, we study the distribution of the function $\omega_k(n)$ over h-full numbers. The definition of h-full numbers enforces that the distribution of $\omega_k(n)$ over h-full numbers is zero for $k \leq h-1$. Thus, we only need to study the case $k \geq h$.

4.1. The first moment of $\omega_k(n)$ over h-full numbers

Proof of Theorem 1.3. Note that

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_k(n) = \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \sum_{\substack{p \mid n \\ p^k \| n}} 1 = \sum_{p \leq x^{1/k}} \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h, p^k \| n}} 1 = \sum_{p \leq x^{1/k}} A_{p,h}(x/p^k), \tag{37}$$

where $A_{p,h}(y)$ is defined in (15). Thus, applying Lemma 2.7 with a single prime p, we have

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_k(n) = \gamma_{0,h} x^{1/h} \sum_{p \leq x^{1/k}} \frac{1 - p^{-1/h}}{p^{k/h}\left(1 - p^{-1/h} + p^{-1}\right)} + O_h \left(x^{1/(h+1)} \sum_{p \leq x^{1/k}} \frac{1}{p^{k/(h+1)}} \right). \tag{38}$$

The above formula for $k = h$ yields

$$\sum_{\substack{n \leq x \\ n \in \mathcal{N}_h}} \omega_h(n) = \gamma_{0,h} x^{1/h} \left(\sum_{p \leq x^{1/h}} \frac{1}{p\left(1 - p^{-1/h} + p^{-1}\right)} \right.$$

$$\left. - \sum_{p \leq x^{1/h}} \frac{1}{p^{1+1/h}\left(1 - p^{-1/h} + p^{-1}\right)} \right)$$

$$+ O_h\left(x^{1/(h+1)} \sum_{p \le x^{1/h}} \frac{1}{p^{h/(h+1)}}\right).$$

By [3, (42)], we have

$$\sum_{p \le x^{1/h}} \frac{1}{p\left(1 - p^{-1/h} + p^{-1}\right)} = \log\log x + B_1 - \log h + \mathcal{L}_h(h+1) - \mathcal{L}_h(2h) + O_h\left(\frac{1}{\log x}\right). \tag{39}$$

Thus, using Lemma 2.2 with $\alpha = h/(h+1)$, and Lemma 2.3 with $y = x^{1/h}$ and $r = h+1$, we obtain

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}_h}} \omega_h(n) = \gamma_{0,h} x^{1/h} \log\log x + \left(B_1 - \log h - \mathcal{L}_h(2h)\right)\gamma_{0,h} x^{1/h} + O_h\left(\frac{x^{1/h}}{\log x}\right).$$

Now, let's consider the case $k > h$. Rewriting (38) for $k > h$ and using Lemma 2.3, we obtain

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}_h}} \omega_k(n) = \gamma_{0,h} x^{1/h}\left(\mathcal{L}_h(k) - \mathcal{L}_h(k+1) + O_{h,k}\left(\frac{1}{x^{\frac{1}{h}-\frac{1}{k}}(\log x)}\right)\right)$$

$$+ O_h\left(x^{1/(h+1)} \sum_{p \le x^{1/k}} \frac{1}{p^{k/(h+1)}}\right).$$

Note that, for $k = h+1$,

$$\sum_{p \le x^{1/k}} \frac{1}{p^{k/(h+1)}} = O(\log\log x)$$

and for $k > h+1$,

$$\sum_{p \le x^{1/k}} \frac{1}{p^{k/(h+1)}} = O_k(1).$$

Combining the above results, we obtain

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}_h}} \omega_{h+1}(n) = \left(\mathcal{L}_h(h+1) - \mathcal{L}_h(h+2)\right)\gamma_{0,h} x^{1/h} + O_h\left(x^{1/(h+1)} \log\log x\right),$$

and for $k > h+1$, we obtain

$$\sum_{\substack{n \le x \\ n \in \mathcal{N}_h}} \omega_k(n) = \left(\mathcal{L}_h(k) - \mathcal{L}_h(k+1)\right)\gamma_{0,h} x^{1/h} + O_{h,k}\left(x^{1/(h+1)}\right).$$

This completes the proof. □

4.2. The second moment of $\omega_k(n)$ over h-full numbers

Proof of Theorem 1.4. Note that, for $k \geq h$, we have

$$\sum_{\substack{n\leq x\\ n\in\mathcal{N}_h}} \omega_k^2(n) = \sum_{\substack{n\leq x\\ n\in\mathcal{N}_h}} \left(\sum_{\substack{p\\ p^k\|n}} 1\right)^2 = \sum_{\substack{n\leq x\\ n\in\mathcal{N}_h}} \omega_k(n) + \sum_{\substack{n\leq x\\ n\in\mathcal{N}_h}} \sum_{\substack{p,q\\ p^k\|n,q^k\|n,p\neq q}} 1, \tag{40}$$

where p and q above denote primes. The first sum on the right-hand side above can be estimated using Theorem 1.3 and for the second sum, we first rewrite the sum, and use Lemma 2.7 with two distinct primes p and q to obtain

$$\begin{aligned}
&\sum_{\substack{n\leq x\\ n\in\mathcal{N}_h}} \sum_{\substack{p,q\\ p^k\|n,q^k\|n,p\neq q}} 1\\
&= \sum_{\substack{p,q\\ p\neq q, pq\leq x^{1/k}}} \sum_{\substack{n\leq x/(pq)^k\\ n\in\mathcal{N}_h\\ (p,n)=(q,n)=1}} 1\\
&= \gamma_{0,h}x^{1/h} \sum_{\substack{p,q\\ pq\leq x^{1/k}}} \frac{1}{p^{k/h}\left(1+\frac{p^{-1}}{1-p^{-1/h}}\right)} \frac{1}{q^{k/h}\left(1+\frac{q^{-1}}{1-q^{-1/h}}\right)}\\
&\quad - \gamma_{0,h}x^{1/h} \sum_{p} \left(\frac{1}{p^{k/h}\left(1+\frac{p^{-1}}{1-p^{-1/h}}\right)}\right)^2 + O_{h,k}\left(\frac{x^{1/2k}}{\log x}\right)\\
&\quad + O_h\left(x^{\frac{1}{h+1}} \sum_{\substack{p,q\\ p\neq q, pq\leq x^{1/h}}} \frac{1}{p^{k/(h+1)}q^{k/(h+1)}}\right).
\end{aligned} \tag{41}$$

For bounding the sum in the error term above, we use Lemma 2.2 and Lemma 2.4 for $k = h$, Lemma 2.5 for $k = h + 1$, and $\sum_{p,q} \frac{1}{(pq)^{k/(h+1)}} = O(1)$ for $k > h + 1$. Thus, we obtain

$$x^{\frac{1}{h+1}} \sum_{\substack{p,q\\ p\neq q, pq\leq x^{1/h}}} \frac{1}{p^{k/(h+1)}q^{k/(h+1)}} \ll_{h,k} \begin{cases} \frac{x^{1/h}\log\log x}{\log x} & \text{if } k = h,\\ x^{1/(h+1)}(\log\log x)^2 & \text{if } k = h+1, \text{ and}\\ x^{1/(h+1)} & \text{if } k > h+1. \end{cases} \tag{42}$$

We study the first sum in the main term in (41) by dividing it into cases. We begin with $k = h$ and estimate the sum

$$\sum_{\substack{p,q \\ pq \le x^{1/h}}} \frac{1-p^{-1/h}}{p\left(1-p^{-1/h}+p^{-1}\right)} \frac{1-q^{-1/h}}{q\left(1-q^{-1/h}+q^{-1}\right)}.$$

Note that

$$\frac{1-p^{-1/h}}{p\left(1-p^{-1/h}+p^{-1}\right)} = \frac{1}{p} - \frac{1}{p^2\left(1-p^{-1/h}+p^{-1}\right)}.$$

Using this, a similar result for a prime q and the symmetry in primes p and q, we expand the previous sum as

$$\begin{aligned} &\sum_{\substack{p,q \\ pq \le x^{1/h}}} \frac{1-p^{-1/h}}{p\left(1-p^{-1/h}+p^{-1}\right)} \frac{1-q^{-1/h}}{q\left(1-q^{-1/h}+q^{-1}\right)} \\ &= \sum_{\substack{p,q \\ pq \le x^{1/h}}} \frac{1}{pq} - 2 \sum_{\substack{p,q \\ pq \le x^{1/h}}} \frac{1}{pq^2\left(1-q^{-1/h}+q^{-1}\right)} \\ &\quad + \sum_{\substack{p,q \\ pq \le x^{1/h}}} \frac{1}{p^2\left(1-p^{-1/h}+p^{-1}\right)} \frac{1}{q^2\left(1-q^{-1/h}+q^{-1}\right)}. \end{aligned}$$

We bound the first sum above using Lemma 2.4. For the second sum, we have

$$\begin{aligned} &\sum_{\substack{p,q \\ pq \le x^{1/h}}} \frac{1}{pq^2\left(1-q^{-1/h}+q^{-1}\right)} \\ &= \sum_{\substack{p \\ p \le x^{1/h}/2}} \frac{1}{p} \sum_{\substack{q \\ q \le x^{1/h}/p}} \frac{1}{q^2\left(1-q^{-1/h}+q^{-1}\right)} \\ &= \mathcal{L}_h(2h)\left(\log\log x + B_1 - \log h\right) + O_h\left(\frac{1}{\log x}\right). \end{aligned}$$

Similarly, for the third sum, we obtain

$$\sum_{\substack{p,q \\ pq \le x^{1/h}}} \frac{1}{p^2\left(1-p^{-1/h}+p^{-1}\right)} \frac{1}{q^2\left(1-q^{-1/h}+q^{-1}\right)} = \left(\mathcal{L}_h(2h)\right)^2 + O_h\left(\frac{\log\log x}{x^{1/h}\log x}\right).$$

Combining the last three results with Lemma 2.4, we obtain

$$\begin{aligned} &\sum_{\substack{p,q \\ pq \le x^{1/h}}} \frac{1-p^{-1/h}}{p\left(1-p^{-1/h}+p^{-1}\right)} \frac{1-q^{-1/h}}{q\left(1-q^{-1/h}+q^{-1}\right)} \\ &= \left(\log\log x - \log h\right)^2 + 2B_1(\log\log x - \log h) + B_1^2 - \zeta(2) \end{aligned}$$

$$- 2\left(\mathcal{L}_h(2h)\left(\log\log x + B_1 - \log h\right)\right) + \left(\mathcal{L}_h(2h)\right)^2 + O_h\left(\frac{\log\log x}{\log x}\right).$$

Combining the above with (40), (41), and (42) for $k = h$ and using Theorem 1.3, we obtain the second moment for $\omega_h(n)$ over h-full numbers.

Now, we focus on the case $k \geq h+1$. Using the definition of $\mathcal{L}_h(r)$ (5), we obtain

$$\sum_{\substack{p,q \\ pq \leq x^{1/k}}} \frac{1 - p^{-1/h}}{p^{k/h}\left(1 - p^{-1/h} + p^{-1}\right)} \frac{1 - q^{-1/h}}{q^{k/h}\left(1 - q^{-1/h} + q^{-1}\right)}$$
$$= \left(\mathcal{L}_h(k) - \mathcal{L}_h(k+1)\right)^2 + O_{h,k}\left(\frac{x^{\frac{1}{k} - \frac{1}{h}} \log\log x}{\log x}\right).$$

Combining the above with (40), (41), and (42) for $k \geq h+1$ and using Theorem 1.3, for $k \geq h+1$, we obtain the required second moments for $\omega_k(n)$ for $k \geq h+1$ over h-full numbers. This completes the proof. □

5. The Erdős-Kac theorems

In this section, we establish the Erdős-Kac theorem for $\omega_1(n)$ over h-free numbers and for $\omega_h(n)$ over h-full numbers. To do this, we employ ideas from [5, Proof of Theorem 1.3]. We prove the following two theorems:

Proof of Theorem 1.5. For an arithmetic function f and a natural number $n \geq 3$, let $r_f(n)$ be the ratio

$$r_f(n) := \frac{f(n) - \log\log n}{\sqrt{\log\log n}}. \tag{43}$$

In this proof, we will be using f to represent ω and ω_1 when necessary. For $a \in \mathbb{R}$ and a subset S of natural numbers, let $S(x)$ denote the set of elements of S up to x, and

$$D(f, S, x, a) := \frac{1}{|S(x)|} \left|\{n \in S(x) \ : \ r_f(n) \leq a\}\right| \tag{44}$$

be the density function for sufficiently large x. Since $\omega_1(n) \leq \omega(n)$, thus $r_{\omega_1}(n) \leq r_\omega(n)$ for all $n \geq 3$. Therefore using $S = \mathcal{S}_h$,

$$D(\omega, \mathcal{S}_h, x, a) \leq D(\omega_1, \mathcal{S}_h, x, a)$$

for all $x \geq 3$. Thus, by the Erdős-Kac theorem for $\omega(n)$ over h-free numbers (see [4, Theorem 1.4]), we have

$$\Phi(a) \leq \liminf_{x \to \infty} D(\omega_1, \mathcal{S}_h, x, a). \tag{45}$$

For any $\epsilon > 0$, we define the set

$$A(\mathcal{S}_h, x, \epsilon) := \left\{ n \in \mathcal{S}_h(x) \ : \ \frac{\omega(n) - \omega_1(n)}{\sqrt{\log\log n}} \le \epsilon \right\}.$$

Let $A^c(\mathcal{S}_h, x, \epsilon)$ denote the complement of $A(\mathcal{S}_h, x, \epsilon)$ inside $\mathcal{S}_h(x)$. Note that

$$r_{\omega_1}(n) = r_\omega(n) + \frac{\omega(n) - \omega_1(n)}{\sqrt{\log\log n}}.$$

Thus, using the definition of $A(\mathcal{S}_h, x, \epsilon)$, we obtain

$$\begin{aligned}
\{n \in \mathcal{S}_h(x) \ : \ r_{\omega_1} \le a\} &= \left\{ n \in \mathcal{S}_h(x) \ : \ r_\omega(n) + \frac{\omega_1(n) - \omega(n)}{\sqrt{\log\log n}} \le a \right\} \\
&= \left\{ n \in A(\mathcal{S}_h, x, \epsilon) \ : \ r_\omega(n) + \frac{\omega_1(n) - \omega(n)}{\sqrt{\log\log n}} \le a \right\} \\
&\quad \cup \left\{ n \in A^c(\mathcal{S}_h, x, \epsilon) \ : \ r_\omega(n) + \frac{\omega_1(n) - \omega(n)}{\sqrt{\log\log n}} \le a \right\} \\
&\subseteq \{n \in \mathcal{S}_h(x) \ : \ r_\omega(n) \le a + \epsilon\} \cup A^c(\mathcal{S}_h, x, \epsilon).
\end{aligned}$$

Then, by the definition of $D(f, \mathcal{S}_h, x, a))$, we have

$$D(\omega_1, \mathcal{S}_h, x, a) \le D(\omega, \mathcal{S}_h, x, a + \epsilon) + \frac{|A^c(\mathcal{S}_h, x, \epsilon)|}{|\mathcal{S}_h(x)|}. \tag{46}$$

We intend to show that the second summand on the right-hand side above goes to 0 as $x \to \infty$. By (6) and Theorem 1.1, we have

$$\sum_{n \in \mathcal{S}_h(x)} (\omega(n) - \omega_1(n)) \ll \frac{x}{\zeta(h)}.$$

Additionally,

$$\begin{aligned}
\sum_{n \in \mathcal{S}_h(x)} (\omega(n) - \omega_1(n)) &\ge \sum_{\substack{\frac{x}{\log x} \le n \le x \\ n \in A^c(\mathcal{S}_h, x, \epsilon)}} (\omega(n) - \omega_1(n)) \\
&> \epsilon \sum_{\substack{\frac{x}{\log x} \le n \le x \\ n \in A^c(\mathcal{S}_h, x, \epsilon)}} \sqrt{\log\log n} \\
&\ge \epsilon \sqrt{\log\log(x/\log x)} \ |\{n \ge x/\log x \ : \ n \in A^c(\mathcal{S}_h, x, \epsilon)\}|.
\end{aligned}$$

The above two results imply

$$|\{n \ge x/\log x \ : \ n \in A^c(\mathcal{S}_h, x, \epsilon)\}| \ll \frac{1}{\epsilon \cdot \zeta(h)} \frac{x}{\sqrt{\log\log(x/\log x)}},$$

where the right-hand side is $o(x)$. Since $|\mathcal{S}_h(x/\log x)|$ is also $o(x)$ and $|\mathcal{S}_h(x)| \gg x/\zeta(h)$, we obtain

$$\lim_{x\to\infty} \frac{|A^c(\mathcal{S}_h, x, \epsilon)|}{|\mathcal{S}_h(x)|} = 0.$$

Finally, taking the limits as $x \to \infty$ on both sides of (46), and using the above with the Erdős-Kac theorem for $\omega(n)$ over h-free numbers, we obtain

$$\limsup_{x\to\infty} D(\omega_1, \mathcal{S}_h, x, a) \leq \Phi(a+\epsilon).$$

Since, $\epsilon > 0$ is arbitrary, combining the above with (45) yields

$$\lim_{x\to\infty} D(\omega_1, \mathcal{S}_h, x, a) = \Phi(a).$$

This completes the proof. □

Proof of Theorem 1.6. Recall the definitions for $r_f(n)$ and $D(f, S, x, a)$ from (43) and (44) respectively. In this proof, we will be using $S = \mathcal{N}_h$ and f to represent ω and ω_h when necessary.

Since $\omega_h(n) \leq \omega(n)$, thus $r_{\omega_h}(n) \leq r_\omega(n)$ for all $n \geq 3$. Therefore using $S = \mathcal{N}_h$,

$$D(\omega, \mathcal{N}_h, x, a) \leq D(\omega_h, \mathcal{N}_h, x, a)$$

for all $x \geq 3$. Thus, by the Erdős-Kac theorem for $\omega(n)$ over h-full numbers (see [4, Theorem 1.5], we have

$$\Phi(a) \leq \liminf_{x\to\infty} D(\omega_h, \mathcal{N}_h, x, a). \tag{47}$$

For any $\epsilon > 0$, we define the set

$$A_h(\mathcal{N}_h, x, \epsilon) := \left\{ n \in \mathcal{N}_h \ : \ \frac{\omega(n) - \omega_h(n)}{\sqrt{\log\log n}} \leq \epsilon \right\}.$$

Let $A_h^c(\mathcal{N}_h, x, \epsilon)$ denote the complement of $A_h(\mathcal{N}_h, x, \epsilon)$ inside $\mathcal{N}_h(x)$. Note that

$$r_{\omega_h}(n) = r_\omega(n) + \frac{\omega(n) - \omega_h(n)}{\sqrt{\log\log n}}.$$

Thus, we obtain

$$\begin{aligned}\{n \in \mathcal{N}_h(x) \ : \ r_{\omega_h} \leq a\} &= \left\{n \in \mathcal{N}_h(x) \ : \ r_\omega(n) + \frac{\omega_h(n) - \omega(n)}{\sqrt{\log\log n}} \leq a\right\} \\ &= \left\{n \in A_h(\mathcal{N}_h, x, \epsilon) \ : \ r_\omega(n) + \frac{\omega_h(n) - \omega(n)}{\sqrt{\log\log n}} \leq a\right\} \\ &\quad \cup \left\{n \in A_h^c(\mathcal{N}_h, x, \epsilon) \ : \ r_\omega(n) + \frac{\omega_h(n) - \omega(n)}{\sqrt{\log\log n}} \leq a\right\} \\ &\subseteq \{n \in \mathcal{N}_h(x) \ : \ r_\omega(n) \leq a + \epsilon\} \cup A_h^c(\mathcal{N}_h, x, \epsilon).\end{aligned}$$

Then, by the definition of $D(f, \mathcal{N}_h, x, a)$, we have

$$D(\omega_h, \mathcal{N}_h, x, a) \leq D(\omega, \mathcal{N}_h, x, a + \epsilon) + \frac{|A_h^c(\mathcal{N}_h, x, \epsilon)|}{|\mathcal{N}_h(x)|}. \tag{48}$$

We again intend to show that the second summand on the right-hand side above goes to 0 as $x \to \infty$. By (7) and Theorem 1.3, we have

$$\sum_{n \in \mathcal{N}_h(x)} (\omega(n) - \omega_h(n)) \ll_h x^{1/h}.$$

Additionally,

$$\begin{aligned}\sum_{n \in \mathcal{N}_h(x)} (\omega(n) - \omega_h(n)) &\geq \sum_{\substack{\frac{x}{\log x} \leq n \leq x \\ n \in A_h^c(\mathcal{N}_h, x, \epsilon)}} (\omega(n) - \omega_h(n)) \\ &> \epsilon \sum_{\substack{\frac{x}{\log x} \leq n \leq x \\ n \in A^c(\mathcal{N}_h, x, \epsilon)}} \sqrt{\log\log n} \\ &\geq \epsilon \sqrt{\log\log(x/\log x)} \ |\{n \geq x/\log x \ : \ n \in A_h^c(\mathcal{N}_h, x, \epsilon)\}|.\end{aligned}$$

The above two results imply

$$|\{n \geq x/\log x \ : \ n \in A_h^c(\mathcal{N}_h, x, \epsilon)\}| \ll_h \frac{1}{\epsilon} \frac{x^{1/h}}{\sqrt{\log\log(x/\log x)}},$$

where the right-hand side is $o(x^{1/h})$. Since the size of the set $|\mathcal{N}_h(x/\log x)|$ is also $o(x^{1/h})$ and $|\mathcal{N}_h(x)| \gg \gamma_{0,h} x^{1/h}$, we obtain

$$\lim_{x \to \infty} \frac{|A_h^c(\mathcal{N}_h, x, \epsilon)|}{|\mathcal{N}_h(x)|} = 0.$$

Finally, taking the limits as $x \to \infty$ on both sides of (48), and using the above with the Erdős-Kac theorem for $\omega(n)$ over h-full numbers, we obtain

$$\limsup_{x \to \infty} D(\omega_h, \mathcal{N}_h, x, a) \leq \Phi(a + \epsilon).$$

Since, $\epsilon > 0$ is arbitrary, combining the above with (47) yields

$$\lim_{x\to\infty} D(\omega_h, \mathcal{N}_h, x, a) = \Phi(a).$$

This completes the proof. □

6. No normal orders

In this section, we prove Theorem 1.7 and Theorem 1.8. The former establishes that the functions $\omega_k(n)$ with $1 < k < h$ do not have normal order over h-free numbers, and the latter proves that $\omega_k(n)$ with $k > h$ do not have normal order over h-full numbers.

Proof of Theorem 1.7. We first assume that $F(n)$ is not identically 0. Then, there exists $n_0 \in \mathbb{N}$ such that $F(n_0) > 0$ for all $n \geq n_0$. For $x > 2$ and $1 < k < h$, let

$$S^h_{0,k}(x) := \{n \in \mathcal{S}_h(x) \ : \ \omega_k(n) = 0\}.$$

Note that

$$S^h_{0,k}(x) \supseteq \mathcal{S}_k(x).$$

Since $|\mathcal{S}_k(x)| \gg x/\zeta(k)$, thus $|S^h_{0,k}(x)| \gg x/\zeta(k)$. In particular, the set of $n \in \mathcal{S}_h(x)$ for which $F(n) > 0$ and $\omega_k(n) = 0$ is not $o(x)$. For all such n, notice that the inequality

$$|\omega_k(n) - F(n)| > \frac{F(n)}{2} \tag{49}$$

is satisfied. Thus, we deduce that $\omega_k(n)$ does not have normal order $F(n)$ when $F(n)$ is not identically zero.

Next, we work with the case when $F(n)$ is identically 0. Let

$$S^h_{1,k}(x) := \{n \in \mathcal{S}_h(x) \ : \ \omega_k(n) = 1\}.$$

Using Lemma 2.6 for the single prime $p = 2$, we deduce

$$\begin{aligned} |S^h_{1,k}(x)| &\geq \sum_{\substack{n\in\mathcal{S}_h(x) \\ p^k \| n \text{ for exactly one prime } p \leq x^{1/k}}} 1 \\ &\geq \sum_{\substack{n\in\mathcal{S}_k(x/2^k) \\ (n,2)=1}} 1 \\ &\gg \frac{2^k - 2^{k-1}}{2^k(2^k-1)} \frac{x}{\zeta(k)}. \end{aligned}$$

Thus, the set of $n \in \mathcal{S}_h(x)$ for which $F(n) > 0$ and $\omega_k(n) = 1$ is not $o(x)$. Also, for all such n, inequality (49) is satisfied. Thus, $\omega_k(n)$ does not have normal order $F(n)$ when $F(n)$ is identically zero. This completes the proof. $\square$

Proof of Theorem 1.8. We first assume that $F(n)$ is not identically 0. Thus, there exists $n_0 \in \mathbb{N}$ such that $F(n_0) > 0$ for all $n \geq n_0$. For $x > 2$ and $k > h$, let

$$N^h_{0,k}(x) := \{n \in \mathcal{N}_h(x) \ : \ \omega_k(n) = 0\}.$$

Note that

$$N^h_{0,k}(x) \supseteq (\mathcal{N}_h \cap \mathcal{S}_k)(x).$$

Moreover, using Lemma 2.8 for the prime $p = 2$, we obtain

$$\begin{aligned} |(\mathcal{N}_h \cap \mathcal{S}_k)(x)| &\geq \sum_{\substack{n \leq x \\ n \in \mathcal{N}_h \cap \mathcal{S}_k, (n,2)=1}} 1 \\ &\gg \frac{1 - 2^{-1/h}}{1 - 2^{-1/h} + 2^{-1} - 2^{-k/h}} \eta_{h,k} x^{1/h}. \end{aligned}$$

Thus, the set of $n \in \mathcal{N}_h(x)$ for which $F(n) > 0$ and $\omega_k(n) = 0$ is not $o(x^{1/h})$. Also, for all such n, inequality (49) is satisfied. Thus, $\omega_k(n)$ does not have normal order $F(n)$ when $F(n)$ is not identically 0.

Next, we work with the case when $F(n)$ is identically 0. Let

$$N^h_{1,k}(x) := \{n \in \mathcal{N}_h(x) \ : \ \omega_k(n) = 1\}.$$

Using Lemma 2.7 for the single prime $p = 2$, we deduce

$$\begin{aligned} |N^h_{1,k}(x)| &\geq \sum_{\substack{n \in \mathcal{N}_h(x) \\ p^k \| n \text{ for exactly one prime } p \leq x^{1/k}}} 1 \\ &\geq \sum_{\substack{n \leq x/2^k \\ n \in \mathcal{N}_h \cap \mathcal{S}_k, (n,2)=1}} 1 \\ &\gg \frac{1 - 2^{-1/h}}{2^{k/h}(1 - 2^{-1/h} + 2^{-1} - 2^{-k/h})} \eta_{h,k} x^{1/h}, \end{aligned}$$

where $\eta_{h,k}$ is given by (16). Therefore, the set of $n \in \mathcal{N}_h(x)$ for which $F(n) > 0$ and $\omega_k(n) = 1$ is not $o(x^{1/h})$. Also, for all such n, inequality (49) is satisfied. Thus, $\omega_k(n)$

does not have normal order $F(n)$ when $F(n)$ is identically zero. This completes the proof. $\square$

In this work, we establish that $\omega_1(n)$ has normal order $\log\log n$ and also satisfies the Erdős-Kac theorem over h-free. Similarly, ω_h has normal order $\log\log n$ and also satisfies the Erdős-Kac theorem over h-full numbers. We also proved that $\omega_k(n)$ with $1 < k < h$ do not have normal order over h-free numbers and $\omega_k(n)$ with $k > h$ do not have normal order over h-full numbers. These results can be generalized to a general number field. The authors have been working on this and will report their findings in a future article. Note that the function field analog of this research has been studied by Gómez and Lalín [7].

Data availability

No data was used for the research described in the article.

Acknowledgment

The authors would like to thank Matilde Lalín for the helpful discussions.

References

[1] K. Alladi, P. Erdős, On an additive arithmetic function, Pac. J. Math. 71 (2) (1977) 275–294.
[2] P. Billingsley, On the central limit theorem for the prime divisor functions, Am. Math. Mon. 76 (1969) 132–139.
[3] S. Das, W. Kuo, Y.-R. Liu, Distribution of $\omega(n)$ over h-free and h-full numbers, 2024, preprint.
[4] S. Das, W. Kuo, Y.-R. Liu, A subset generalization of the Erdős-Kac theorem over number fields with applications, 2024, preprint.
[5] E. Elma, Y.-R. Liu, Number of prime factors with a given multiplicity, Can. Math. Bull. 65 (1) (2022) 253–269.
[6] P. Erdős, M. Kac, The Gaussian law of errors in the theory of additive number theoretic functions, Am. J. Math. 62 (1940) 738–742.
[7] J.A. Gómez, M. Lalín, Prime factors with given multiplicity in h-free and h-full polynomials over function fields, 2023, Pre-print.
[8] A. Granville, K. Soundararajan, Sieving and the Erdős-Kac theorem, in: Equidistribution in Number Theory, An Introduction, in: NATO Sci. Ser. II Math. Phys. Chem., vol. 237, Springer, Dordrecht, 2007, pp. 15–27.
[9] H. Halberstam, On the distribution of additive number-theoretic functions, J. Lond. Math. Soc. 30 (1955) 43–53.
[10] H. Halberstam, On the distribution of additive number-theoretic functions. II, J. Lond. Math. Soc. 31 (1956) 1–14.
[11] H. Halberstam, On the distribution of additive number-theoretic functions. III, J. Lond. Math. Soc. 31 (1956) 14–27.
[12] G.H. Hardy, S. Ramanujan, The normal number of prime factors of a number n [Quart. J. Math. 48 (1917) 76–92], in: Collected Papers of Srinivasa Ramanujan, AMS Chelsea Publ., Providence, RI, 2000, pp. 262–275.
[13] G.H. Hardy, E.M. Wright, An Introduction to the Theory of Numbers, sixth edition, Oxford University Press, Oxford, 2008, Revised by D.R. Heath-Brown and J.H. Silverman, with a foreword by Andrew Wiles.
[14] R. Jakimczuk, M. Lalín, Sums of $\omega(n)$ and $\Omega(n)$ over the k-free parts and k-full parts of some particular sequences, Integers 22 (2022) A113.

[15] M. Lalín, Z. Zhang, The number of prime factors in h-free and h-full polynomials over function fields, Publ. Math. (Debr.) 104 (3–4) (2024) 377–421.
[16] M.R. Murty, Problems in Analytic Number Theory, second edition, Graduate Texts in Mathematics, vol. 206, Springer, New York, 2008, Readings in Mathematics.